PHYS 301
HOMEWORK #3

Due : 6 Feb. 2013

For this homework assignment, you may use Mathematica to help compute integrals; if you do, submit your Mathematica output with your homework.

For problems 1 - 3 find the complex Fourier series for the functions indicated. Calculate the Fourier coefficients and write out the first three non - zero pairs of terms of the series.

For complex Fourier series, we will use the following results :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

$$\cos nx = \frac{e^{inx}+e^{-inx}}{2} \qquad \sin nx = \frac{e^{inx}-e^{-inx}}{2i}$$

---

1. $f(x) = x^2,\ -\pi < x < \pi$

Solution : Using the standard equations we find :

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,e^{-inx}\,dx = \frac{2(-1)^n}{n^2}$$

Our first few coefficients become :

$$c_1 = 2(-1)^1 = -2 \qquad c_{-1} = 2(-1)^{-1} = -2$$

$$c_2 = \frac{2(-1)^2}{2^2} = \frac{1}{2} \qquad c_2 = \frac{2(-1)^{-2}}{(-2)^2} = \frac{1}{2}$$

$$c_3 = \frac{2(-1)^3}{3^2} = \frac{-2}{9} \qquad c_{-3} = \frac{2(-1)^{-3}}{(-3)^2} = \frac{-2}{9}$$

The first three pairs of non zero terms are :

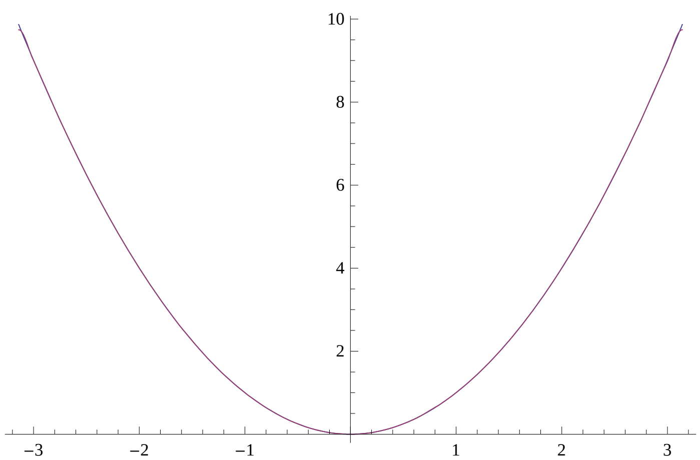
$$f(x) = \frac{\pi^2}{3} - 2\left[\left(e^{i x} + e^{-i x}\right) - \frac{\left(e^{2 i x} + e^{-2 i x}\right)}{4} + \frac{\left(e^{3 i x} + e^{-3 i x}\right)}{9} - \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \ldots\right]$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos n x}{n^2}$$

Verifying with Mathematica :

```
Plot[{x², π² / 3 + 4 Sum[(-1)ⁿ Cos[n x] / n², {n, 1, 31}]}, {x, -π, π}]
```



---

2. $f(x) = \text{Abs}[x], \ -\pi < x < \pi$

Solution : We can use symmetry arguments to find c_0 (but not c_n since $e^{-i n x}$ is neither even nor odd):

$$c_0 = \frac{2}{2\pi} \int_0^{\pi} x \, dx = \frac{\pi}{2}$$

$$c_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\, e^{-i n x} dx + \int_0^{\pi} x\, e^{-i n x} dx\right] = \frac{1}{2\pi}\left[\frac{-1 + e^{i n \pi}(1 - i n \pi) - 1 + e^{-i n \pi}(1 + i n \pi)}{n^2}\right]$$

For integer values of n, $e^{\pm i n \pi} = \cos(n\pi) = (-1)^n$, this expression becomes:

$$c_n = \frac{1}{2\pi} \frac{(-2 + 2(-1)^n)}{n^2} = \begin{cases} 0, & n \text{ even} \\ -2/\pi n^2, & n \text{ odd} \end{cases}$$

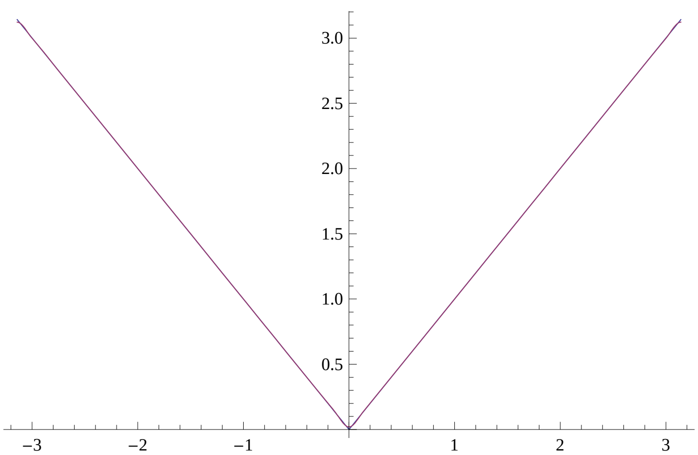
The first few terms of the Fourier expansion are then :

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left[\left(e^{ix} + e^{-ix}\right) + \frac{\left(e^{3ix} + e^{-3ix}\right)}{3^2} + \frac{\left(e^{5ix} + e^{-5ix}\right)}{5^2} + \ldots\right] =$$

$$\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

Verifying graphically :

```
Plot[{Abs[x], π / 2 - (4 / π) Sum[Cos[n x] / n^2, {n, 1, 31, 2}]}, {x, -π, π}]
```



---

3. $f(x) = e^{ax}$, $-\pi < x < \pi$ (a is a real number)

Solution : The choice of f (x) in this case is neither even nor odd; integrating we find :

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{\sinh a\pi}{a\pi}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\,dx = \frac{1}{2\pi(a-in)}\,e^{ax}\,e^{-inx}\Big|_{-\pi}^{\pi} =$$

$$\frac{1}{2\pi(a-in)}\left[e^{a\pi}\,e^{-in\pi} - e^{-a\pi}\,e^{+in\pi}\right] = \frac{2(-1)^n}{2\pi(a-in)}\sinh(a\pi)$$

In the final step, we made use of the definition of the sinh function and the earlier result that $e^{in\pi}$ and $e^{-in\pi} = \cos(n\pi) = (-1)^n$. Multiplying numerator and denominator by the complex conjugate of the denominator, and separating into real and imaginary parts we get:

$$c_n = \frac{(-1)^n (a+in)\sinh(a\pi)}{\pi\left(a^2+n^2\right)} = \frac{(-1)^n a\sinh(a\pi)}{\pi\left(a^2+n^2\right)} + i\,\frac{(-1)^n\, n\sinh(a\pi)}{\pi\left(a^2+n^2\right)}$$

The real part of the coefficients will yield cos terms, and the imaginary part will yield the sin terms; this Fourier series will have both cos and sin terms as you might expect from a function that is neither even nor odd. Computing coefficients, and remembering that n will be both positive and negative, and also that i = -1/i, we get :

$$c_1 = \frac{-a \sinh a\pi}{\pi\left(a^2+1\right)} + \frac{\sinh a\pi}{\pi\left(a^2+1\right) i}; \quad c_{-1} = -\frac{a \sinh a\pi}{\pi\left(a^2+1\right)} - \frac{\sinh a\pi}{\pi\left(a^2+1\right) i}$$

$$c_2 = +\frac{a \sinh (a\pi)}{\pi\left(a^2+4\right)} - \frac{2 \sinh a\pi}{\pi\left(a^2+4\right) i}; \; c_{-2} = \frac{a \sinh (a\pi)}{\pi\left(a^2+4\right)} + \frac{2 \sinh a\pi}{\pi\left(a^2+n^2\right) i}$$

$$c_3 = -\frac{a \sinh a\pi}{\pi\left(a^2+9\right)} + \frac{3 \sinh (a\pi)}{\pi\left(a^2+9\right) i}; \quad c_{-3} = -\frac{a \sinh a\pi}{\pi\left(a^2+9\right)} - \frac{3 \sinh (a\pi)}{\pi\left(a^2+9\right) i}$$

Now, remember that our Fourier series terms will be :

$$c_0 + c_1 e^{i x} + c_{-1} e^{-i x} + c_2 e^{2 i x} + c_{-2} e^{-2 i x} + \ldots$$

The real parts of $c_1 e^{i x} + c_{-1} e^{-i x}$ will yield:

$$-\frac{a \sinh a\pi}{\pi\left(a^2+1\right)}\left(e^{i x}+e^{-i x}\right) = -\frac{2 a \sinh a\pi \cos x}{\pi\left(a^2+1\right)}$$

The imaginary parts of this sum yield :

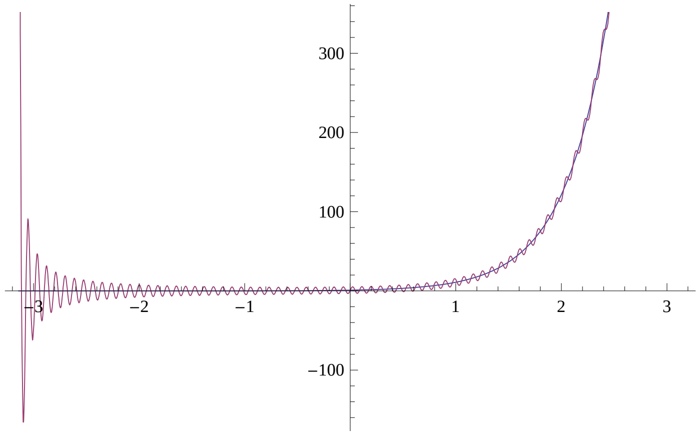
$$\frac{\sinh a\pi}{\pi\left(a^2+1\right)}\left(\frac{e^{i x}-e^{-i x}}{i}\right) = \frac{2 \sinh a\pi \sin x}{\pi\left(a^2+1\right)}$$

Following this pattern, we obtain for our Fourier expansion :

$$f(x) = \frac{\sinh a\pi}{a\pi} - \frac{\sinh a\pi}{\pi}\left[\frac{2 a \cos x}{a^2+1} - \frac{2 a \cos 2x}{a^2+4} + \frac{2 a \cos 3x}{a^2+9} - \frac{2 \sin x}{a^2+1} + \frac{4 \sin x}{a^2+4} - \frac{6 \sin 3x}{a^2+9} + ..\right]$$

Verifying :

```
Clear[a]
a = 2.4;
Plot[{Exp[a x], Sinh[a π] / (a π) + (2 Sinh[a π] / π) (a Sum[(-1)^n Cos[n x] / (a^2 + n^2), {n, 1, 71}] -
      Sum[(-1)^n n Sin[n x] / (a^2 + n^2), {n, 1, 71}])}, {x, -π, π}]
```



Remarkably, the series converges to the function.

For problems 4 - 7, find the trigonometric Fourier series for the functions indicated; be sure to take into account the interval specified. Calculate the Fourier coefficients and write out the first three non - zero terms.

For the next set of problems, we will make frequent use of the following expressions :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$$

$$a_0 = \frac{1}{L}\int_{\text{lower limit}}^{\text{upper limit}} f(x)\,dx \quad \{a_n,\ b_n\} = \frac{1}{L}\int_{ll}^{ul} f(x)\{\cos(n\pi x/L),\ \sin\{n\pi x/L\}\}\,dx$$

---

4. $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < 2\pi \end{cases}$

Solution : The total length of the interval, 2 L, is 2 π, so L = π and we compute :

$$a_0 = \frac{1}{\pi}\int_0^{\pi/2} 1 \cdot dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi/2} \cos(n\pi x/\pi)\,dx = \frac{1}{\pi n}\sin(n x)\Big|_0^{\pi/2} =$$

$$\frac{1}{n\pi}\sin(n\pi/2) = \begin{cases} 0, & n \text{ even} \\ 1/n\pi, & n = 1, 5, 9, \ldots \\ -1/n\pi, & n = 3, 7, 11, \ldots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi/2} \sin(n x)\, dx = \frac{-1}{\pi n} \cos(n x)\Big|_0^{\pi/2} =$$

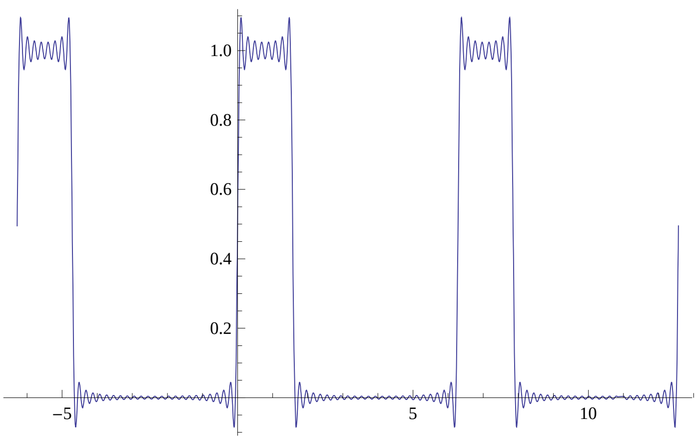
$$\frac{1}{\pi n}(1 - \cos(n\pi/2)) = \begin{cases} 1/n\pi, & \text{n odd} \\ 2/n\pi, & n = 2, 6, 10, \ldots \\ 0, & n = 4, 8, 12, \ldots \end{cases}$$

And our Fourier series can be written :

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots\right) + \frac{1}{\pi}\left(\sin x + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right)$$

Verifying via Mathematica :

```
Plot[(1 / 4) + (1 / π) Sum[Sin[n π / 2] Cos[n x] / n, {n, 1, 31}] +
  (1 / π) Sum[(1 - Cos[n π / 2]) Sin[n x] / n, {n, 1, 31}], {x, -2 π, 4 π}]
```



---

5. $f(x) = \sin(x/2),\ 0 < x < 2\pi$

Solution :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \sin(x/2)\, dx = \frac{-2}{\pi}\cos(x/2)\Big|_0^{2\pi} = \frac{-2}{\pi}(\cos\pi - 1) = \frac{4}{\pi}$$

Making use of sin/cos addition formulae, we can find the other Fourier coefficients :

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sin(x/2)\cos(n x)\, dx = \left(\frac{1}{\pi}\right)\left[\frac{\cos((2n-1)x/2)}{2n-1} - \frac{\cos((2n+1)x/2)}{2n+1}\right]\Big|_0^{2\pi}$$

When the cos terms are evaluated at x = 2 π, we get terms of the form cos ((2 n ± 1) π). For integer values of n, 2 n ± 1 is always odd, so these terms are always - 1; when evaluated at x = 0, the cos terms equal 1, so our coefficients are :

$$a_n = \frac{1}{\pi}\left(\frac{-2}{2n-1} - \frac{-2}{2n+1}\right) = \frac{4}{\left(\pi\left(1-4n^2\right)\right)}$$

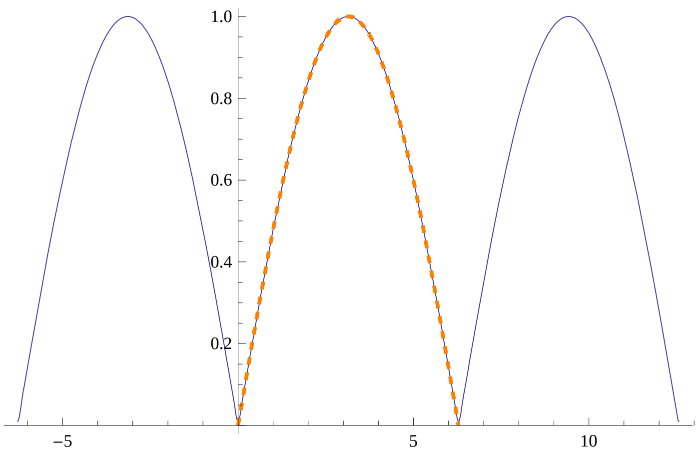
$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sin(x/2)\sin(n x)\,dx = \left(\frac{1}{\pi}\right)\left[\frac{\sin(1/2 - n)x}{2n-1} - \frac{\sin(1/2+n)x}{2n+1}\right]\Big|_0^{2\pi} = 0$$

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos x}{1-4} + \frac{\cos 2x}{1-4\cdot 2^2} + \frac{\cos 3x}{1-4\cdot 3^2} + \ldots\right]$$

Verifying by plotting the Fourier series from -2 π to 6π on the same axis as f(x) (which is shown in orange) :

In[381]:=
```
g1 = Plot[(2 / π) + (4 / π) Sum[Cos[n x] / (1 - 4 n^2), {n, 1, 31}], {x, -2 π, 4 π}];
g2 = Plot[Sin[x / 2], {x, 0, 2 π}, PlotStyle → {Thick, Dashed, Orange}];
Show[g1, g2]
```

Out[383]=



---

6. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$

Solution : Finding the Fourier coefficients for an interval where 2 L = 4 and L = 2 :

$$a_0 = \frac{1}{2}\int_0^2 x\,dx = 1$$

$$a_n = \frac{1}{2}\int_0^2 x\cos(n\pi x/2)\,dx = \frac{4\left(-1+(-1)^n\right)}{2n^2\pi^2} = \begin{cases} 0, & n\text{ even} \\ -4\big/n^2\pi^2, & n\text{ odd}\end{cases}$$

$$b_n = \frac{1}{2}\int_0^2 x\sin(n\pi x/2)\,dx = \frac{-2(-1)^n}{n\pi}$$

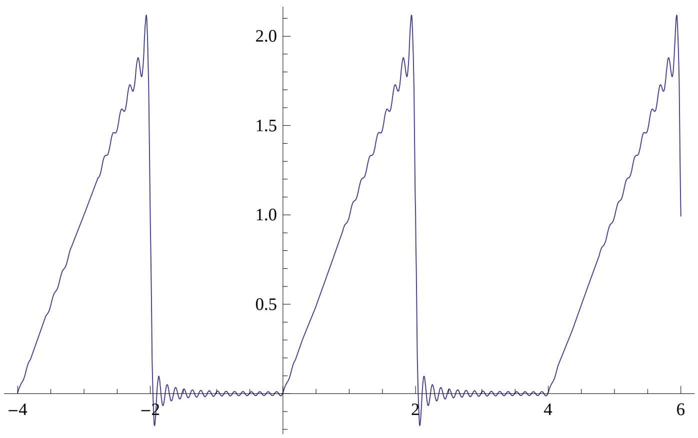
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos(\pi x/2) + \frac{\cos(3\pi x/2)}{9} + \frac{\cos(5\pi x/2)}{25} + \ldots\right] + \frac{2}{\pi}\left[\sin(\pi x/2)\right.$$

$$-\frac{\sin(2\pi x/2)}{2}+\frac{\sin(3\pi x/2)}{3}+\ldots\Big]$$

Verifying :

```
In[337]:= Plot[(1 / 2) - (4 / π^2) Sum[Cos[n π x / 2] / n^2, {n, 1, 31, 2}] -
   (2 / π) Sum[(-1)^n Sin[n π x / 2] / n, {n, 1, 31}], {x, -4, 6}]
```

Out[337]=



---

7. $f(x) = \begin{cases} 0, & -3 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

Solution : Here is a case where our interval is not symmetric across the origin; this will not effect our evaluation of Fourier coefficients. The length of the interval is 4, so L = 2 and :

$$a_0=\frac{1}{2}\int_0^1 1\cdot dx = \frac{1}{2}$$

$$a_n=\frac{1}{2}\int_0^1 1\cdot\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{2n\pi}\sin(n\pi x/2)\Big|_0^1 =$$

$$\frac{1}{n\pi}\sin(n\pi/2)=\begin{cases} 0, & \text{n even} \\ 1/n\pi, & n=1,\ 5,\ 9,\ \ldots \\ -1/n\pi, & n=3,\ 7,\ 11,\ \ldots \end{cases}$$

$$b_n=\frac{1}{2}\int_0^1 1\cdot\sin\left(\frac{n\pi x}{2}\right)dx = \frac{-2}{2n\pi}\cos(n\pi x/2)\Big|_0^1 =$$

$$\frac{1}{n\pi}(1-\cos(n\pi/2)) = \begin{cases} 1/n\pi, & \text{n odd} \\ 2/n\pi, & n=2,\ 6,\ 10,\ \ldots \\ 0, & n=4,\ 8,\ 12 \end{cases}$$

$$f(x) = \frac{1}{4}+\frac{1}{\pi}\left[\cos(\pi x/2)-\frac{\cos(3\pi x/2)}{3}+\frac{\cos(5\pi x/2)}{5}+\ldots\right]+$$
$$\frac{1}{\pi}\left[\sin(\pi x/2)+\frac{2\sin(2\pi x/2)}{2}+\frac{\sin(3\pi x/2)}{3}+\frac{\sin(5\pi x/2}{5}+\frac{2\sin(6\pi x/2)}{6}+\ldots\right)\Big]$$

---

8. Use Mathematica to plot your Fourier Series for the interval - 7 < x < 5. Submit your output

with your homework.

Solution : Using the coefficients from above we get :

In[368]:=
```
Plot[(1 / 4) + (1 / π) Sum[Sin[n π / 2] Cos[n π x / 2] / n, {n, 1, 31}] +
  (1 / π) Sum[(1 - Cos[n π / 2]) Sin[n π x / 2] / n, {n, 1, 31}], {x, -7, 5}]
```

Out[368]=