PHYS 301
HOMEWORK #4

Due : 15 February 2013

On this homework, you may use Mathematica for help in evaluation of indefinite integrals only. You must do all other work (including evaluation of coefficients) by hand. If you use Mathematica, please submit your output with your homework, and make sure your name is on your homework. As with all homeworks this semester, make sure your solutions show your work clearly.

1. In class, we outlined the solution to the wave equation as illustrated in problem 9 - 23 of Chapter 7. Compute the Fourier coefficients for this problem and verify that :

$$a_0 = a_n = 0$$

$$b_n = \begin{cases} 0, & n \text{ even} \\ 8h/(n^2\pi^2), & n = 1, 5, 9, \ldots \\ -8h/(n^2\pi^2), & n = 3, 7, 11, \ldots \end{cases}$$

Solution : In class we showed that the string could be represented by the functional form :

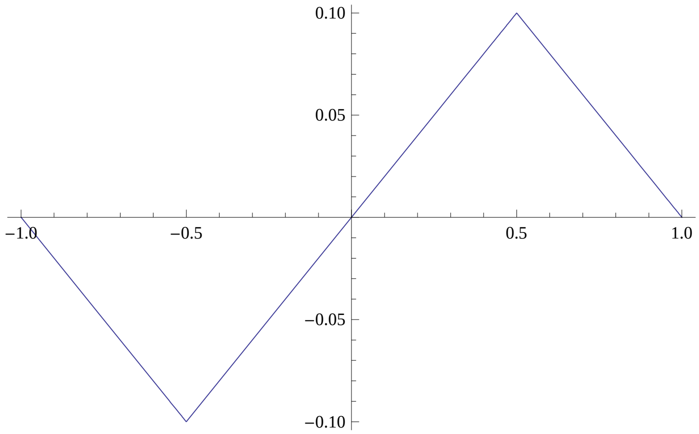
$$f(x) = \begin{cases} 2hx/L, & 0 < x < L/2 \\ 2h(1 - x/L), & L/2 < x < L \end{cases}$$

Since we need to make the odd extension for this function, we will find the Fourier series for g (x) on (-L, L) with

$$g(x) = \begin{cases} -2h - 2hx/L, & -L < x < -L/2 \\ 2hx/L, & -L/2 < x < L/2 \\ 2h(1 - x/L), & L/2 < x < L \end{cases}$$

Plotting this function, we represent the string as :

```
h = 0.1; L = 1;
Plot[Piecewise[{{-2 h - 2 h x / L, -L < x < -L / 2},
   {2 h x / L, -L / 2 < x < L / 2}, {2 h (1 - x / L), L / 2 < x < L}}], {x, -L, L}]
```



Because the function g(x) is odd, we can appeal to symmetry to determine that $a_0 = a_n = 0$. Symmetry allows us to write:

$$b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Our task is to determine the values of :

$$b_n = \frac{2}{L}\left(\int_0^{L/2}\frac{2\,h\,x}{L}\sin(n\pi x/L)\,dx + \int_{L/2}^{L} 2\,h\,(1 - x/L)\sin(n\pi x/L)\,dx\right)$$

We can always find the expression for a definite integral by grind and find, let' s see if we can find a more elegant way to evaluate this integral. First, we can rewrite this expression as :

$$\frac{2}{L}\left(\int_0^{L/2}\frac{2\,h\,x}{L}\sin(n\pi x/L)\,dx - \int_{L/2}^{L}\frac{2\,h\,x}{L}\sin(n\pi x/L)\,dx + \int_{L/2}^{L} 2\,h\sin(n\pi x/L)\,dx\right)$$

Notice that the first two integrals have the same integrand, albeit different limits of integration. Let' s define a function :

$$K(x) = \int (2\,h\,x/L)\sin(n\pi x/L) = 2\,h\left(\frac{-x\cos(n\pi x/L)}{n\pi} + \frac{L\sin(n\pi x/L)}{n^2\pi^2}\right)$$

(this is just the result you will get from solving the indefinite integral via Mathematica).

Then our final expression for the b coefficients will be :

$$b_n = \frac{2}{L}\left(K(0, L/2, L) + \int_{L/2}^{L} 2\,h\sin(n\pi x/L)\,dx\right) \tag{1}$$

First, let' s evaluate K (x) at x = 0, L/2, and L. We can write the value of the evaluated definite integral as :

$$K(0, L/2, L) = K(L/2) - K(0) - (K(L) - K(L/2)) = 2K(L/2) - K(0) - K(L)$$

evaluating at the limits, we get :

$K(0, L/2, L) =$

$$2\times 2h\left(\frac{\frac{-L}{2}\cos(n\pi/2)}{n\pi} + \frac{L\sin(n\pi/2)}{n^2\pi^2}\right) - 0 - 2h\left(\frac{-L\cos(n\pi)}{n\pi} + \frac{L^2\sin(n\pi)}{n^2\pi^2}\right)$$

It should be easy to see that K = 0 at L = 0, and the last term goes to zero since sin $(n\pi)$ is zero for integer values of n. The last integral in eq. (1) is easily evaluated as :

$$2h\int_{L/2}^{L}\sin(n\pi x/L)\,dx = -\frac{2hL}{n\pi}(\cos(n\pi) - \cos(n\pi/2))$$

Adding all these terms, we get finally :

$$b_n = \frac{2}{L}\left(\frac{-2hL\cos(n\pi/2)}{n\pi} + \frac{4hL\sin(n\pi/2)}{n^2\pi^2} + \frac{2hL\cos(n\pi)}{n\pi} - \frac{2hL\cos(n\pi)}{n\pi} + \frac{2hL\cos(n\pi/2)}{n\pi}\right)$$

Notice that all the cos terms cancel, leaving us with :

$$b_n = \frac{8h\sin(n\pi/2)}{n^2\pi^2} = \begin{cases} 0, & n \text{ even} \\ 8h/n^2\pi^2, & n = 1 \text{ Mod } 4 \\ -8h/n^2\pi^2, & n = 3 \text{ Mod } 4 \end{cases}$$

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2. Consider the function :

$$f(x) = \begin{cases} 10, & 0 < x < 10 \\ 20, & 10 < x < 20 \end{cases}$$

Find the appropriate Fourier series for this function if you a) extend this function to make it an odd function, and b) extend it to make an even function. You will see boundary conditions like this again in our study of partial differential equations (section 13.2).

Solutions : **For the odd extension** we must extend f to x = -20 making our function odd on (-20, 20). Therefore L = 20 for this case, and using symmetry we know that all the a coefficients will be zero, and we can find :

$$b_n = \frac{2}{20}\left(\int_0^{10} 10 \sin(n\pi x/20)\,dx + \int_{10}^{20} 20 \sin(n\pi x/L)\,dx\right) =$$

$$\frac{2}{20}\cdot 10\left(\frac{-20}{n\pi}\cos(n\pi x/20)\Big|_0^{10}\right) + \frac{2}{20}\cdot 20\left(\frac{-20}{n\pi}\cos(n\pi x/20)\Big|_{10}^{20}\right) =$$

$$\frac{20}{n\pi}(1-\cos(n\pi/2)) + \frac{40}{n\pi}(\cos(n\pi/2) - \cos(n\pi))$$

The Fourier sin series is then :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/20) = \frac{20}{\pi}\sum_{n=1}^{\infty}\left[(1-\cos(n\pi/2) + 2(\cos(n\pi/2) - \cos(n\pi))\right]$$
$$\sin(n\pi x/20)/n$$

Verifying :

In[51]:=
```
Clear[n, x]
Plot[(20 / π) Sum[((1 - Cos[n π / 2] + 2 (Cos[n π / 2] - (-1)^n))) Sin[n π x / 20] / n, {n, 1, 51}],
 {x, -20, 20}]
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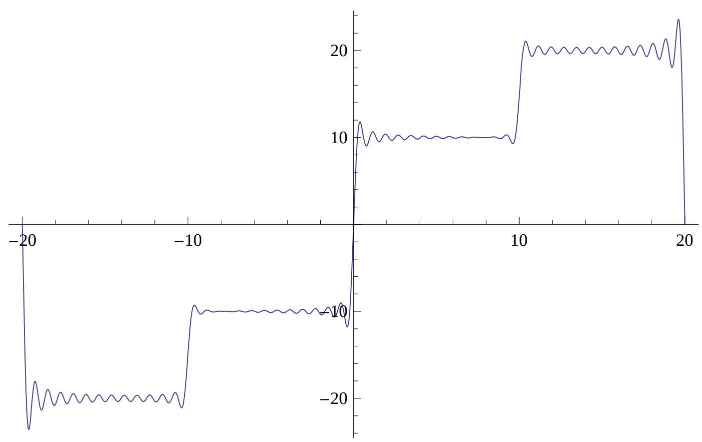
Out[52]=


For the even extension, we have that the b coefficients are zero, and that :

$$a_0 = \frac{2}{20}\left[\int_0^{10} 10\,dx + \int_{10}^{20} 20\,dx\right] = 30$$

$$a_n = \frac{2}{20}\int_0^{10} 10\,\mathrm{Cos}[n\pi x/20]\,dx + \frac{2}{20}\int_{10}^{20} 20\cos[n\pi x/20]\,dx =$$

$$\frac{2}{20}\cdot 10\cdot\frac{20}{n\pi}\sin(n\pi x/20)\Big|_0^{10} + \frac{2}{20}\cdot 20\cdot\frac{20}{n\pi}\sin(n\pi x/20)\Big|_{10}^{20} =$$

$$\frac{20}{n\pi}\sin(n\pi/2)+\frac{40}{n\pi}(\sin n\pi-\sin(n\pi/2)) = \frac{-20}{n\pi}\sin(n\pi/2)$$

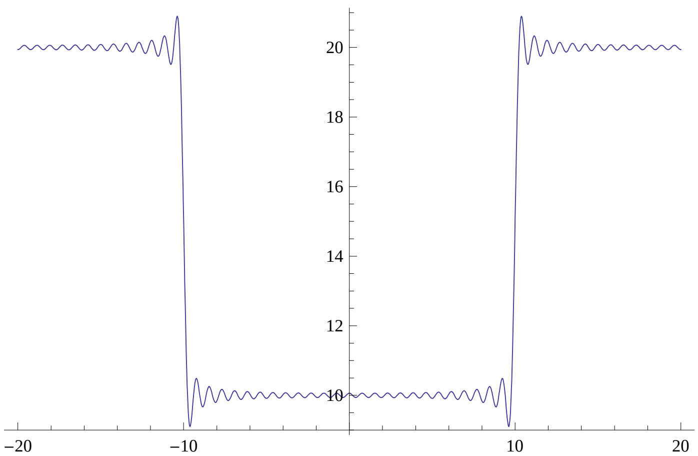
Thus, the Fourier cos series is :

$$f(x) = \frac{30}{2}-\frac{20}{\pi}\sum_{n=1}^{\infty}\sin(n\pi/2)\cos(n\pi x/20)$$

Verifying :

In[54]:=
```
Plot[15 - (20 / π) Sum[Sin[n π / 2] Cos[n π x / 20] / n, {n, 1, 51}], {x, -20, 20}]
```

Out[54]=



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3. Find the Fourier coefficients for f (x) = x on (-1, 1), and use these values with Parseval' s theorem to determine :

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Solution : Parseval' s theorem states :

$$\text{average value of } [f(x)]^2 = \left(\frac{a_0}{2}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}a_n{}^2+\frac{1}{2}\sum_{n=1}^{\infty}b_n{}^2$$

In this case :

$$\text{average value of } x^2 = \frac{\int_{-1}^{1}x^2\,dx}{2} = \frac{1}{3}$$

Since x is odd on (-1, 1), all the a coefficients are zero, and we can find the b coefficients from :

$$b_n = \frac{1}{1}\int_{-1}^{1}x\sin(n\pi x)\,dx = \frac{-2(-1)^n}{n\pi}$$

such that $b_n{}^2 = \dfrac{4}{n^2\pi^2}$, then

$$\text{average of } x^2 = \frac{1}{3} = \frac{1}{2}\sum_{n=1}^{\infty} b_n{}^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2\,\pi^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

We can verify :

```
Sum[1 / n^2, {n, 1, ∞}]
```

$\frac{\pi^2}{6}$

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4. Find the Fourier coefficients for f (x) = x^2 on (-1/2, 1/2) and use these values with Parseval's theorem to determine:

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Solution : Using Parseval' s theorem again, we find the average value of $\left(x^2\right)^2 = x^4$ on (-1/2, 1/2):

$$\text{average value of } f^2 = \frac{\int_{-1/2}^{1/2} x^4\,dx}{1} = \frac{1}{80}$$

Since the function is even, the b coefficients are zero, and we find the a coefficients :

$$a_0 = \frac{2}{1/2}\int_0^{1/2} x^2\,dx = \frac{1}{6}$$

$$a_n = \frac{2}{1/2}\int_0^{1/2} x^2 \cos(2\,n\,\pi\,x)\,dx = \frac{(-1)^n}{n^2\,\pi^2}$$

By Parseval' s Theorem we have :

$$\frac{1}{80} = \left(\frac{1}{12}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4\,\pi^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Verifying :

```
Sum[1 / n^4, {n, 1, ∞}]
```

$\frac{\pi^4}{90}$

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5. Consider a vector A that is a function of time whose magnitude is constant. Show that the vector is perpendicular to its time derivative, i.e.,

$$\frac{d\mathbf{A}(t)}{dt}\cdot\mathbf{A}(t) = 0$$

(It is not correct to say that since the magnitude is a constant, d**A**/dt is always zero.)

Describe a physical situation in which the time derivative of a vector is perpendicular to the vector.

Solution : The magnitude of $\mathbf{A}(t)$ is $\mathbf{A}(t) \cdot \mathbf{A}(t)$, and since its magnitude is constant, we know the time derivative of a constant is zero, therefore :

$$\frac{d}{dt}(\mathbf{A}(t) \cdot \mathbf{A}(t)) = \mathbf{A}(t) \cdot \frac{d\,\mathbf{A}(t)}{dt} + \frac{d\mathbf{A}(t)}{dt} \cdot \mathbf{A}(t) = 2\,\mathbf{A}(t) \cdot \frac{d\mathbf{A}(t)}{dt} = 0$$

Therefore :

$\mathbf{A}(t) \cdot \dfrac{d\,\mathbf{A}(t)}{dt} = 0$ and A is perpendicular to its time derivative.

As hinted in class, a particle moving in a circular orbit has a constant magnitude of distance from the origin, and the velocity vector (the time derivative of displacement) is always perpendicular to the radius vector.

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6. Evaluate :

a) $\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{im}$ b) $\epsilon_{ijk}\,\delta_{jk}$

Where the δ are Kronecker deltas and ϵ is the Levi - Civita permutation tensor.

Solutions :

a) Repeatedly contracting Kronecker deltas, we get :

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{im} = \delta_{ik}\,\delta_{km}\,\delta_{im} = \delta_{im}\,\delta_{im} = \delta_{im}\,\delta_{mi} = \delta_{ii} = 3$$

b) We will consider two possibilities, j = k and j ≠ k; these two possibilities cover all situations. If j = k, the ϵ term is zero since there will be two equal indices, and if one term in a product is zero, the product is zero. If j ≠ k, then the Kronecker delta term is zero and the product is zero. In either event, the product is zero.

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7. Write short Mathematica programs that will find the product of the first 20 Fibonacci numbers. One program should compute this using a Do loop, another using a For statement, and the third using a While statement.

Solutions :

In the Do loop, we use the variable "prod"; in the For statement we use "prod1" and in the While statement we declare the variable as "prod2" :

```
Clear[fib, prod]
fib[1] = 1; fib[2] = 1; prod = 1;
fib[n_] := fib[n] = fib[n - 1] + fib[n - 2]
Do[prod = prod fib[n], {n, 1, 20}]
Print["The product of the first 20 Fibonacci numbers is = ", prod]
```

```
The product of the first 20 Fibonacci numbers is =
 9 692 987 370 815 489 224 102 512 784 450 560 000
```

```
prod1 = 1;
For[i = 1, i < 21, i++, prod1 = prod1 fib[i]]
Print["The product of the first 20 Fibonacci numbers is = ", prod1]
```

The product of the first 20 Fibonacci numbers is = 9 692 987 370 815 489 224 102 512 784 450 560 000

```
prod2 = 1; i = 1; While[i < 21, prod2 = prod2 fib[i]; i++]
Print["The product of the first 20 Fibonacci numbers is = ", prod2]
```

The product of the first 20 Fibonacci numbers is = 9 692 987 370 815 489 224 102 512 784 450 560 000