PHYS 301
HOMEWORK #6

Due : 13 March 2013

In this homework set, **r** represents the position vector and r represents the scalar magnitude of the position vector. In questions 2, 3, 4 and 6, use Einstein summation notation (no credit will be given for proofs using term-by-term component expansion). You may use *Mathematica* to verify your answers, but you must show complete work for all problems.

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1. Evaluate $\nabla \cdot (r^3\,\mathbf{r})$ (your answer will be in terms of r)

Solution : We use the result from class that :

$\nabla\cdot(c\,\mathbf{v}) = c\,\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla c$ where c is a constant and v is a vector. Applied here, we get :

$$\nabla \cdot (r^3\,\mathbf{r}) = r^3\,\nabla\cdot\mathbf{r} + \mathbf{r}\cdot\nabla r^3$$

The divergence of **r** is just 3 (since $\nabla\cdot\mathbf{r} = \partial x_i / \partial x_i = \partial x/\partial x + \partial y/\partial y + \partial z/\partial z = 1+1+1$). So that the first term on the right is just $3\,r^3$. To find the gradient of r^3, we start by writing:

$$r^3 = (x^2+y^2+z^2)^{3/2}$$

so that :

$$\nabla r^3 = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2}\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{3/2}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{3/2}\,\hat{\mathbf{z}} =$$

$$\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2\,x\,\hat{\mathbf{x}} + \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2\,y\,\hat{\mathbf{y}} + \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2\,z\,\hat{\mathbf{z}} =$$

$$3\,(x^2+y^2+z^2)^{1/2}\left[x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\right] = 3\,r\,\mathbf{r}$$

Therefore, the second term on the right becomes $\mathbf{r}\cdot 3\,r\,\mathbf{r} = 3\,r^3$ (remember that $\mathbf{r}\cdot\mathbf{r} = r^2$), so the entire expression is equal to $6\,r^3$.

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2. Prove $\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}$ where f and g are scalars.

Solution : We start by writing the identity in summation notation and apply the quotient rule :

$$\nabla\left(\frac{f}{g}\right) \to \frac{\partial}{\partial x_i}\left(\frac{f}{g}\right) = \frac{g\,\frac{\partial f}{\partial x_i} - f\,\frac{\partial g}{\partial x_i}}{g^2}$$

Since $\partial\phi/\partial x_i$ is summation notation for $\nabla\phi$, the equation above is just:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

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3. Prove $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\,\mathbf{A} - \mathbf{B}\,(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}\,(\nabla\cdot\mathbf{B})$

Solution : We start by defining **C = A × B** which we can represent as :

$$C_i = \epsilon_{ijk} A_j B_k$$

which produces the ith component of the cross product; we can represent the curl of C as :

$$\nabla\times\mathbf{C} \to \epsilon_{mni}\frac{\partial}{\partial x_n} C_i = \epsilon_{mni}\frac{\partial}{\partial x_n}\left(\epsilon_{ijk} A_j B_k\right) = \epsilon_{mni}\,\epsilon_{ijk}\frac{\partial}{\partial x_n}\left(A_j B_k\right)$$

This will give us the m^{th} component of the curl of **A × B**. At this point we realize we have a product of ϵ with a repeated index; we permute the first ϵ cyclically so we can make use of the ϵ - δ relationship :

$$\epsilon_{mni}\,\epsilon_{ijk}\frac{\partial}{\partial x_n}\left(A_j B_k\right) = \epsilon_{imn}\,\epsilon_{ijk}\frac{\partial}{\partial x_n}\left(A_j B_k\right) = \left(\delta_{mj}\,\delta_{nk} - \delta_{mk}\,\delta_{nj}\right)\frac{\partial}{\partial x_n}\left(A_j B_k\right)$$

Now, if we apply the product rule we see how we obtain the four terms :

$$\delta_{mj}\,\delta_{nk}\,B_k\frac{\partial}{\partial x_n}A_j - \delta_{mk}\,\delta_{nj}\,B_k\frac{\partial}{\partial x_n}A_j + \delta_{mj}\,\delta_{nk}\,A_j\frac{\partial}{\partial x_n}B_k - \delta_{mk}\,\delta_{nj}\,A_j\frac{\partial}{\partial x_n}B_k \qquad (1)$$

Now, for the first term in eq. (1), m = j and k = n, so the first term becomes :

$$B_n\frac{\partial}{\partial x_n}A_m$$

Notice that the repeated index is n, so the dot product is between B and the del operator, yielding the term **(B·∇) A.**

In the second term, m = k and n = j, yielding :

$$-B_m\frac{\partial}{\partial x_n}A_n = -\mathbf{B}\,(\nabla\cdot\mathbf{A})$$

since the repeated index is n and the dot is between ∇ and A. In the third term, m = j and n = k producing :

$$A_m\frac{\partial}{\partial x_n}B_n = \mathbf{A}\,(\nabla\cdot\mathbf{B})$$

Finally, since m=k and n=j in the fourth term, it becomes :

$$-A_n \frac{\partial}{\partial x_n} B_m = -(\mathbf{A}\cdot\nabla)\,\mathbf{B}$$

Summing all four terms produces the identity:

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\,\mathbf{A} - \mathbf{B}\,(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}\,(\nabla\cdot\mathbf{B})$$

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4. A vector field is called irrotational if its curl is zero. A vector field is called solenoidal if its divergence is zero. If **A** and **B** are irrotational, prove that **A × B** is solenoidal.

Solution : We are required to show that :

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = 0$$

We use summation notation to show that :

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) \to \frac{\partial}{\partial x_i}\left(\epsilon_{ijk} A_j B_k\right) = \epsilon_{ijk}\frac{\partial}{\partial x_i}\left(A_j B_k\right) = \epsilon_{ijk}\left[A_j \frac{\partial}{\partial x_i} B_k + B_k \frac{\partial}{\partial x_i} A_j\right] =$$

$$A_j\,\epsilon_{ijk}\frac{\partial}{\partial x_i} B_k + B_j\,\epsilon_{ijk}\frac{\partial}{\partial x_i} A_j = -\mathbf{A}\cdot(\nabla\times\mathbf{B}) + \mathbf{B}\cdot(\nabla\times\mathbf{A})$$

Since both **A** and **B** are irrotational, their curls are both zero, so the entire expression is zero and $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = 0$.

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5. Evaluate $\nabla^2\left[\nabla\cdot\left(\mathbf{r}/r^2\right)\right]$

Solution :

In Cartesian coordinates: We begin by evaluating the divergence, using the same identity used in problem 1 :

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right) = \frac{1}{r^2}\,\nabla\cdot\mathbf{r} + \mathbf{r}\cdot\nabla\left(\frac{1}{r^2}\right)$$

The first term on the right is $3\,r^{-2}$. Finding the gradient in the second term:

$$\nabla\left(\frac{1}{r^2}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2+z^2}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2+z^2}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{x^2+y^2+z^2}\right)\hat{\mathbf{z}} =$$

$$\frac{-2\,x}{\left(x^2+y^2+z^2\right)}\,\hat{\mathbf{x}} + \frac{-2\,y}{\left(x^2+y^2+z^2\right)}\,\hat{\mathbf{y}} + \frac{-2\,z}{\left(x^2+y^2+z^2\right)}\,\hat{\mathbf{z}} = \frac{-2\,\mathbf{r}}{r^4}$$

Therefore, the value of the divergence is :

$$\frac{3}{r^2} + \mathbf{r} \cdot \left(-2 \frac{\mathbf{r}}{r^4}\right) = \frac{1}{r^2} = \frac{1}{x^2 + y^2 + z^2}$$

The Laplacian of this term is :

$$\nabla^2\left(\frac{1}{x^2 + y^2 + z^2}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{x^2 + y^2 + z^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{x^2 + y^2 + z^2}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{x^2 + y^2 + z^2}\right)$$

Let' s consider the second derivative with respect to x; once we have this term we know the other two terms will be of an analogous form. The second derivative with respect to x gives us :

$$\frac{8 x^2}{\left(x^2 + y^2 + z^2\right)^3} - \frac{2}{\left(x^2 + y^2 + z^2\right)^2}$$

The second derivative terms in y and z will yield similar terms so that the total Laplacian is :

$$\frac{8 x^2}{\left(x^2 + y^2 + z^2\right)^3} + \frac{8 y^2}{\left(x^2 + y^2 + z^2\right)^3} + \frac{8 z^2}{\left(x^2 + y^2 + z^2\right)^3} - \frac{6}{\left(x^2 + y^2 + z^2\right)^2} =$$

$$\frac{8\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^3} - \frac{6}{\left(x^2 + y^2 + z^2\right)^2} = \frac{8 r^2}{r^6} - \frac{6}{r^4} = \frac{2}{r^4}$$

In spherical coordinates :

I am not expecting anyone to do this in spherical coordinates (since we will show how to do vector calc in spherical coords after break), but this is a good example to show why we transform from one coordinate system to another. In spherical coordinates, the divergence is given by :

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, V_\theta) + \frac{1}{r \sin\theta} \frac{\partial V_\phi}{\partial \phi}$$

where r, θ, ϕ are the radial direction, polar angle and azimuthal angle respectively, and the sub-scripts of V indicate the component of the vector V in that direction. Now let’s find the divergence of our vector, $r^{-2}\,\mathbf{r}$. There is only a radial component (the θ and ϕ components are zero). To find the radial component, remember that the unit vector in the **r** direction is:

$$\hat{r} = \frac{\mathbf{r}}{|r|} \Rightarrow \mathbf{r} = |r|\,\hat{\mathbf{r}}$$

so that we can write this vector as :

$$\frac{\mathbf{r}}{r^2} = \frac{|r|\,\hat{\mathbf{r}}}{r^2} = \frac{\hat{\mathbf{r}}}{r}$$

and so the radial component of our vector is just 1/r. Substituting this into the divergence equation gives us :

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{1}{r} \right) = \frac{1}{r^2}$$

Now, we take the Laplacian of this result; in spherical coordinates, the Laplacian of a scalar f is written as :

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \left(\frac{\partial^2 f}{\partial\phi^2} \right)$$

In our case, f is simply $1/r^2$, so we need only compute :

$$\nabla^2 \left(\frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial(1/r^2)}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{-2}{r^3} \right) = \frac{1}{r^2} \cdot \left(\frac{2}{r^2} \right) = \frac{2}{r^4} \quad \text{as we found before.}$$

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6. Prove curl grad $\phi = 0$ for all scalar fields ϕ (i.e., $\nabla \times \nabla\phi = 0$)

Solution : The crux of this proof is recalling that we can interchange the order of differentiation without changing the result, i.e.,:

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}$$

Our identity becomes :

$$\nabla \times \nabla\phi = \epsilon_{ijk} \left(\frac{\partial}{\partial x_j} \frac{\partial\phi}{\partial x_k} \right)$$

Since order of differentiation does not matter, we can write this as :

$$\nabla \times \nabla\phi = \epsilon_{ijk} \left(\frac{\partial}{\partial x_j} \frac{\partial\phi}{\partial x_k} \right) = \epsilon_{ijk} \left(\frac{\partial}{\partial x_k} \frac{\partial\phi}{\partial x_j} \right) \tag{2}$$

In the last expression in the equation above, changing order of differentiation changes the parity of the permutation tensor, so that we separately know (pay close attention to the subscripts and notice that the j and k subscripts have been reversed) :

$$\epsilon_{ijk} \left(\frac{\partial}{\partial x_k} \frac{\partial\phi}{\partial x_j} \right) = -\,\epsilon_{ijk} \left(\frac{\partial}{\partial x_j} \frac{\partial\phi}{\partial x_k} \right) \tag{3}$$

If we combine the results of eqs. (2) and (3), we obtain :

$$\epsilon_{ijk} \left(\frac{\partial}{\partial x_j} \frac{\partial\phi}{\partial x_k} \right) = -\,\epsilon_{ijk} \left(\frac{\partial}{\partial x_j} \frac{\partial\phi}{\partial x_k} \right)$$

In other words, we have an expression identical to its negative; the only way this can always be true is if the expression is always zero. The result is general; the curl of the gradient of any scalar function is zero.