PHYS 301
HOMEWORK #7--Solutions

You may use Mathematica to verify results, but must show all work by hand.

1. For the vector :

$$\mathbf{v} = x^2\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + x\,y\,z\,\hat{\mathbf{z}}$$

find the value of the line integral

$$\int_C \mathbf{v}\cdot d\mathbf{l}$$

a) along the path that goes from the origin to (1, 1, 1) in three steps : from the origin to (1, 0, 0), then to (1, 1, 0) then to (1, 1, 1)

Solution : We will need to compute three separate line integrals (one for each of the discrete steps of the path). Each integral will have the form :

$$\int_C \mathbf{v}\cdot d\mathbf{l} = \int \left(v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}}\right)\cdot\left(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}\right) = \int \left(v_x\,dx + v_y\,dy + v_z\,dz\right)$$

For this vector function, this becomes :

$$\int \left(x^2\,dx + y\,dy + x\,y\,z\,dz\right)$$

From (0, 0, 0) to (1, 0, 0) : Along the first segment from the origin to (1, 0, 0) we have that y = z = 0 and dy = dz = 0, so the line integral for this segment is simply :

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

From (1, 0, 0) to (1, 1, 0) : Along the second segment, dx = dz = 0 and the line integral is :

$$\int_0^1 y\,dy = \frac{1}{2}$$

From (1, 1, 0) to (1, 1, 1) : For the final segment, x = y = 1, dx = dy = 0 so we have :

$$\int_0^1 z\,dz = \frac{1}{2}$$

The total line integral is the sum of these 3 segments, and is equal to 4/3.

b) along the straight line path from the origin to (1, 1, 1)

Solution : Here, we can parameterize our function as :

$$x = y = z = t$$
$$dx = dy = dz = dt$$

and our limits of integration are from t = 0 to t = 1. So our line integral becomes :

$$\int\left(x^2\,dx + y\,dy + x\,y\,z\,dz\right) = \int_0^1\left(t^2\,dt + t\,dt + t^3\,dt\right) = \frac{1}{3} + \frac{1}{2} + \frac{1}{4} = \frac{13}{12}$$

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2. If **r** is the position vector, find the value of the line integral

$$\oint \mathbf{r}\cdot d\mathbf{r}$$

along the circle defined by

$$x^2 + y^2 = a^2$$

Solution 1 : Recall that the position vector in the x - y plane is :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

and therefore

$$d\mathbf{r} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}$$

The integral becomes :

$$\int \mathbf{r}\cdot d\mathbf{r} = \int(x\,dx + y\,dy)$$

Since our path is along the circle of radius a, we can use the parameterization :

$$x = a\cos\theta \quad dx = -a\sin\theta\,d\theta$$
$$y = a\sin\theta \quad dy = a\cos\theta\,d\theta$$

With this parameterization, we obtain :

$$\int_C(x\,dx + y\,dy) = \int_0^{2\pi}\left(-a^2\sin\theta\cos\theta\,d\theta + a^2\sin\theta\cos\theta\,d\theta\right) = 0$$

Solution 2 : We can use Stokes' Theorem to note that :

$$\int_C \mathbf{r}\cdot d\mathbf{r} = \int_S(\nabla\times\mathbf{r})\cdot\mathbf{n}\,da$$

It is easy to show that the curl of the position vector is zero, hence the line integral is zero. In a similar vein, we know the line integral of a conservative field around a closed loop is zero. Knowing that $\nabla\times\mathbf{r} = 0$ allows this conclusion.

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3. If **r** is the position vector, find the value of

$$\int_S \mathbf{r}\cdot d\mathbf{a}$$

where S is the surface of the unit cube with corners at the origin and (1, 1, 1).

Solution 1 : The more elegant (and better) solution to this problem is to use the Divergence Theorem and write :

$$\int_S \mathbf{r} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{r})\, d\tau$$

Since we know that div r = 3, the integral is simply $\int 3\, d\tau = 3\,V$ where V is the volume of the cube. The volume is trivially 1, so the value of the integral is simply 3.

Solution 2 : We could perform the surface integral explicitly integrating the flux of **r** through each of the six faces of the cube.

For the surface in the y - z plane at x = 1, the outward normal is in the + $\hat{x}$ direction, so the integral is

$$\int \mathbf{r} \cdot \hat{\mathbf{x}}\, da = \int_0^1 \int_0^1 x\, dy\, dz = 1$$

For the opposite face, the integral is the same as above (except for a minus sign since the outward normal is in the - x direction). However,x = 0 at the "back" face of the cube so the value of the integral is zero.

Each pair of faces will follow the same pattern; one face will yield a surface integral equal to 1 and the opposite side will yield an integral equal to zero. Summing over the three dimensions of the box, we once again obtain a value of 3 for the entire surface integral.

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4. For the vector

$$\mathbf{v} = 4\,y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}} + 2\,z\,\hat{\mathbf{z}}$$

evaluate $\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$

over the hemisphere represented by the upper half plane of

$$x^2 + y^2 + z^2 = a^2$$

(this is the upper half of the sphere of radius a centered on the origin).

Solution : We can solve this either by direct integration or by using Stokes' Theorem, using the circle at the base of the hemisphere to define our contour. Then, we have

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l} = \oint \left(\mathbf{v}_x\, dx + v_y\, dy + v_z\, dz\right)$$

with the contour being the circle of radius a centered on the origin. We can eliminate the dz term since z = dz = 0 in the x - y plane. Using the same set of parameterizations that we used in problem 2 and obtain :

$$v_x = 4\,y = 4\,a \sin\theta$$
$$v_y = x = a \cos\theta$$

$$dx = -a \sin\theta \, d\theta$$
$$dy = a \cos\theta \, d\theta$$

Substituting these into the integal yields :

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l} = \oint (v_x \, dx + v_y \, dy) = \int_0^{2\pi} \left(-4 a^2 \sin^2\theta + a^2 \cos^2\theta\right) d\theta$$

We have shown many time previously in the course that

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

so that the value of this integral is then - $3\pi a^2$.

Now let' s compute the surface integral directly. The normal to the base of the hemisphere is in the z direction, so we need to find the z component of the curl :

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y & x & 2z \end{vmatrix} \Rightarrow \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y} 4y\right)\hat{\mathbf{z}} = -3\hat{\mathbf{z}}$$

The surface integral becomes :

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_S -3\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} \, da = -3\pi a^2 \text{ as before.}$$

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5. For the vector :

$$\mathbf{F} = 3xy\,\hat{\mathbf{x}} - y^2\,\hat{\mathbf{y}}$$

evaluate :

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

along the path $y = 2x^2$ from the origin to (1, 2)

Solution : Our parameterization will be :

$$x = t \quad dx = dt$$
$$y = 2t^2 \quad dy = 4t \, dt$$

The line integral becomes :

$$\int_C (F_x \, dx + F_y \, dy) = \int_0^1 \left(3t \cdot 2t^2 \, dt - 4t^4 \cdot 4t \, dt\right) = \int_0^1 \left(6t^3 - 16t^5\right) dt = \frac{6}{4} - \frac{16}{6} = \frac{-7}{6}$$