PHYS 301
HOMEWORK #8
Solutions

For questions 1 and 2, you may use Mathematica to verify intermediate results but you must show all work by hand.

1. The transformation equations for parabolic coordinates are :

$$x = u\,v\cos\phi$$
$$y = u\,v\sin\phi$$
$$z = \frac{1}{2}\left(u^2 - v^2\right)$$

Verify that this is an orthogonal transformation (there are a number of ways to do this, you only need to show this once). Find expressions for the scale factors and the unit vectors $\hat{\boldsymbol{u}}, \hat{\mathbf{v}}, \hat{\boldsymbol{\phi}}$.

Solution : We begin by finding the scale factors by determining expressions for dx, dy and dz.

$$dx = v\cos\phi\,du + u\cos\phi\,dv - u\,v\sin\phi\,d\phi$$
$$dy = v\sin\phi\,du + u\sin\phi\,dv + u\,v\cos\phi\,d\phi$$
$$dz = u\,du - v\,dv$$

We know that the quantity

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

must be the same in all coordinate systems, therefore we have :

$$(ds)^2 = (v\cos\phi\,du + u\cos\phi\,dv - u\,v\sin\phi\,d\phi)^2 + (v\sin\phi\,du + u\sin\phi\,dv + u\,v\cos\phi\,d\phi)^2 + (u\,du - v\,dv)^2 =$$

$$(du)^2\left(v^2\cos^2\phi + v^2\sin^2\phi + u^2\right) + (dv)^2\left(u^2\cos^2\phi + u^2\sin^2\phi + v^2\right) + (d\phi)^2\left(u^2 v^2\left(\sin^2\phi + \cos^2\phi\right)\right) =$$

$$\left(u^2 + v^2\right)(du)^2 + \left(u^2 + v^2\right)(dv)^2 + \left(u^2 v^2\right)(d\phi)^2$$

All the mixed terms (du dv, du dϕ, and dv dϕ) sum to zero; this is one way of showing this is an orthogonal transformation. The scale factors are given by :

$$h_u = \sqrt{u^2 + v^2} \quad h_v = \sqrt{u^2 + v^2} \quad h_\phi = u\,v$$

To find the unit vectors we first write the position vector :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = u\,v\cos\phi\,\hat{\mathbf{x}} + u\,v\sin\phi\,\hat{\mathbf{y}} + \frac{1}{2}\left(u^2 - v^2\right)\hat{\mathbf{z}}$$

and then compute for each unit vector the quantity :

$$\hat{\mathbf{q}}_{\mathbf{i}} = \frac{\partial \mathbf{r} / \partial q_i}{|\partial \mathbf{r} / \partial q_i|}$$

then,

$$\hat{\mathbf{u}} = \frac{\partial \mathbf{r} / \partial u}{|\partial \mathbf{r} / \partial u|} = \frac{v \cos\phi\, \hat{\mathbf{x}} + v \sin\theta\, \hat{\mathbf{y}} + u\, \hat{\mathbf{z}}}{\sqrt{u^2 + v^2}}$$

$$\hat{\mathbf{v}} = \frac{\partial \mathbf{r} / \partial v}{|\partial \mathbf{r} / \partial v|} = \frac{u \cos\phi\, \hat{\mathbf{x}} + u \sin\phi\, \hat{\mathbf{y}} - v\, \hat{\mathbf{z}}}{\sqrt{u^2 + v^2}}$$

$$\hat{\boldsymbol{\phi}} = \frac{\partial \mathbf{r} / \partial \phi}{|\partial \mathbf{r} / \partial \phi|} = \frac{-u\, v \sin\phi\, \hat{\mathbf{x}} + u\, v \cos\phi\, \hat{\mathbf{y}}}{u\, v} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}$$

We can also show this is an orthogonal transformation by taking all possible dyadic dot products :

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{v^2 \cos^2\phi + v^2 \sin^2\phi + u^2}{u^2 + v^2} = 1$$

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{v}} = \frac{u^2 \cos^2\phi + u^2 \sin^2\phi + v^2}{u^2 + v^2} = 1$$

$$\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \sin^2\phi + \cos^2\phi = 1$$

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \frac{u\, v \cos^2\phi + u\, v \sin^2\phi - u\, v}{u^2 + v^2} = 0$$

$$\hat{\mathbf{u}} \cdot \hat{\boldsymbol{\phi}} = \frac{-u\, v^2 \cos\phi \sin\phi + u\, v^2 \cos\phi \sin\phi}{u\, v \sqrt{u^2 + v^2}} = 0$$

$$\hat{\mathbf{v}} \cdot \hat{\boldsymbol{\phi}} = \frac{-v\, u^2 \cos\phi \sin\phi + v\, u^2 \cos\phi \sin\phi}{u\, v \sqrt{u^2 + v^2}} = 0$$

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2. The transformation equations for the spherical coordinate system are :

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

where r is the distance from the origin, θ is the polar angle (measured down from the north pole) and

ϕ is the azimuthal angle.

a) Find the scale factors and unit vectors for the spherical coordinate system. (20)
b) Express the position vector completely in terms of spherical polar coordinates. (10)
c) Find the expressions for velocity and acceleration in spherical polar coordinates. (30)

Solution : For a complete solution to this, see the other solution set posted for this homework.

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3. Use numerical methods to solve the differential equation :

$$\frac{dy}{dx} = y \text{ subject to the condition } y(0) = 1$$

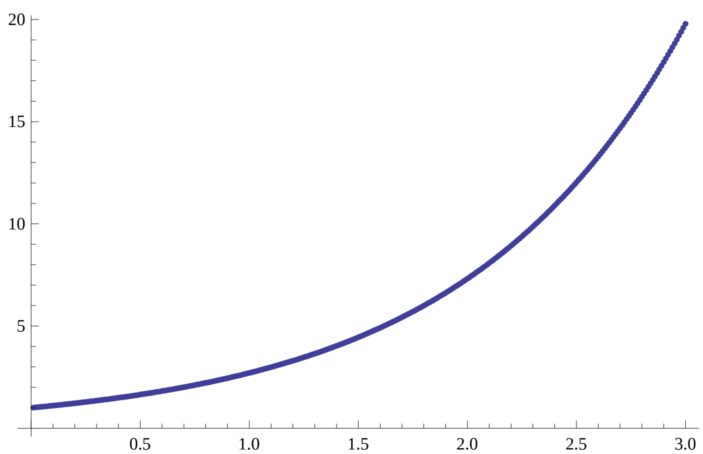
Write a short Mathematica program to solve this differential equation (using Euler' s method). Your output should include your program and a plot of your tabulated results from x = 0 to x = 3. (Do not use DSolve or NDSolve or any other Mathematica library function that directly solves differential equations). (25)

Solution : I present the program below, with comments after the plots.

In[1]:=

```
Clear[x, y, h]
x[0] = 0; y[0] = 1; h = 0.01;
x[n_] := x[n] = x[n - 1] + h
y[n_] := y[n] = y[n - 1] + h y[n - 1]
ListPlot[Table[{x[n], y[n]}, {n, 300}]]
```

Out[5]=



The first line clears all variables we will use; the second line establishes initial conditions and sets the step size to be 0.01. The third line instructs the program to increment the independent variable, x, by 0.01 units, so that we evaluate our function at x = 0, x = 0.01, x = 0.02, etc. The next line utilizes Euler' s method. Euler' s method is based on the definition of the derivative :

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow f(x+h) = f(x) + h f'(x)$$

In other words, Euler' s method computes the next value of f by adding f’ to the previously computed to value of f. In our case, the derivative is just y, so our evaluation of y[n] becomes:

$$y[n] = y[n-1] + \text{step size} * \text{derivative} = y[n-1] + h\, y[n-1]$$

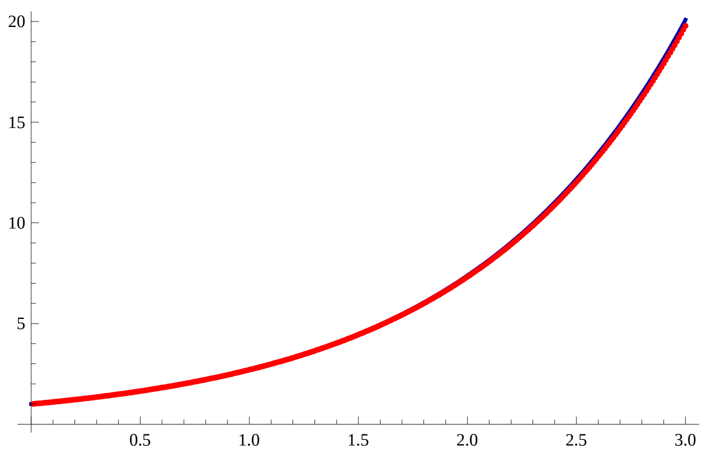
Let' s see if our program matches the answer we know we should get. The given differential equation can be solved trivially to yield

$$y = e^x$$

We will plot the graphs of the analytic and numerical solutions on the same set of axes :

In[36]:=
```
g1 = Plot[Exp[x], {x, 0, 3}, PlotStyle → {Darker[Blue], Thick}];
g2 = ListPlot[Table[{n h, y[n]}, {n, 300}], PlotStyle → {Red, Thin, Dashed}];
Show[g1, g2]
```

Out[38]=



If you study this graph carefully, you will see that there are indeed two curves (the blue curve representing the analytic solution and the red curve the numerical solution) that overlap so closely it is difficult to differentiate them. Make sure you understand why the ListPlot here uses nh on the x - axis.