PHYS 301
HOMEWORK #10

Due : 28 March 2014

1. Use Einstein summation notation to prove that $\nabla\times(\nabla\phi) = 0$ for all scalar ϕ. (You will need to figure out how to use summation notation to reflect the interchangeability of partial derivatives).

Solution : We begin by writing the identity in summation notation :

$$\nabla\times(\nabla\phi) = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k}$$

Since we can interchange the order of partial differentiation, we can write :

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k} = \epsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial\phi}{\partial x_j} \tag{1}$$

Now, recall the properties of the permutation tensor. In the expression on the LHS of eq. (1), the order is cyclic; in the expression on the right hand side, the order is anti - cyclic since we exchanged the indices k and j. This means that the RHS is the negative of the LHS, or

$$-\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k} = \epsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial\phi}{\partial x_j} \Rightarrow \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k} = -\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k}$$

The only way an expression is always the negative of itself is if the expression is zero always.

2. Start on this one as soon as you can. Consider the transformation equations for spherical polar coordinates :

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

where r is the scalar distance from the origin, θ is the polar angle (measured down from the north pole), and ϕ is the azimuthal angle, measured counterclockwise from the + x axis.

a) Find the scale factors for the spherical polar coordinates. (10)
b) Find the unit vectors. (10)
c) Express the position vector r solely in terms of spherical polar coordinates. (10)
d) Find the expressions for velocity and acceleration in spherical polars. (30)

You can probably find all these solutions on line, but I urge you (as I do in all cases) not to use them. Working these through for yourself will take time, but will be a critical help to your understanding of how transformations work, and will also provide a solid foundation for more advanced courses, especially theoretical mechanics.