PHYS 301
HOMEWORK #11
Solutions

1. Begin by taking the differentials of x, y and z :

$$dx = v\cos\phi\, du + u\cos\phi\, dv - u\, v\sin\phi\, d\phi$$

$$dy = v\sin\phi\, du + u\sin\phi\, dv + u\, v\cos\phi\, d\phi$$

$$dz = u\, du - v\, dv$$

$$\begin{aligned}(dx)^2 + (dy)^2 + (dz)^2 = {} & \left(v^2\cos^2\phi + v^2\sin^2\phi + u^2\right)(du)^2 + \left(u^2\cos^2\phi + u^2\sin^2\phi + v^2\right)(dv)^2 + \\ & \left(u^2 v^2\sin^2\phi + u^2 v^2\cos^2\phi\right)(d\phi)^2 \\ & + 2\left(u\, v\cos^2\phi\, u\, v\sin^2\phi - u\, v\right)(du\, dv) \\ & + 2\left(-u\, v^2\cos\phi\sin\phi + u\, v^2\sin\phi\cos\phi\right)(du\, d\phi) \\ & + 2\left(-u\, v^2\cos\phi\sin\phi + u\, v^2\sin\phi\cos\phi\right)(dv\, d\phi)\end{aligned}$$

We can see that all the cross term (du dv, du dϕ and dv dϕ) sum to zero, leaving only terms involving perfect squares, one indication of an orthogonal transformation.

We can apply basic algebra and trig to the results above to obtain :

$$h_u = \sqrt{u^2 + v^2} \qquad h_v = \sqrt{u^2 + v^2} \qquad h_\phi = u\, v$$

The unit vectors are found by writing the position vector as :

$$r = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = u\, v\cos\phi\,\hat{\mathbf{x}} + u\, v\sin\phi\,\hat{\mathbf{y}} + \frac{1}{2}\left(u^2 - v^2\right)\hat{\mathbf{z}}$$

$$\hat{u} = \frac{\partial\mathbf{r}/\partial u}{|\partial\mathbf{r}/\partial u|} = \frac{v\cos\phi\,\hat{\mathbf{x}} + v\sin\phi\,\hat{\mathbf{y}} + u\,\hat{\mathbf{z}}}{\sqrt{u^2 + v^2}}$$

$$\hat{\mathbf{v}} = \left|\frac{\partial\mathbf{r}/\partial v}{\partial r/\partial v}\right| = \frac{u\cos\phi\,\hat{\mathbf{x}} + u\sin\phi\,\hat{\mathbf{y}} - v\,\hat{\mathbf{z}}}{\sqrt{u^2 + v^2}}$$

$$\hat{\phi} = \left|\frac{\partial\mathbf{r}/\partial\phi}{\partial\mathbf{r}/\partial\phi}\right| = \frac{-u\, v\sin\phi\,\hat{\mathbf{x}} + u\, v\cos\phi\,\hat{\mathbf{y}}}{u\, v}$$

You can also confirm that the transformation is orthogonal by verifying that all dot products satisfy

$$\hat{\mathbf{e}}_{\mathbf{i}}\,\hat{\mathbf{e}}_{\mathbf{j}} = \delta_{ij}$$

2. We begin with the expression for acceleration in spherical polar coordinates, given on the last page of the solutions for the last homework :

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta\right)\hat{\mathbf{r}}$$

$$+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta\right)\hat{\boldsymbol{\theta}}$$

$$+\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta\right)\hat{\boldsymbol{\phi}}$$

Since each of the rings has a fixed radius, all terms involving $\dot{r}$ and $\ddot{r}$ will be zero. Similarly, since all angular velocities are constant, all second derivatives of ϕ and θ are zero. The pink ring is rotating only in the ϕ direction, while the blue and green rings are rotating in both θ and ϕ. Therefore, the acceleration expression for the pink ring will not have $\dot{\theta}$ terms, while the expressions for the other rings will. Thus, we can write the acceleration of a particle on the pink ring as:

$$\mathbf{a} = -r\,\omega_a^2 \sin^2\theta\,\hat{\mathbf{r}} - r\,\omega_a^2 \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}$$

where ω_a is the angular velocity about the polar axis. If we call ω_b the angular velocity about the equatorial axis, we have for the blue and green rings:

$$\mathbf{a} = \left(-r\,\omega_b^2 - r\,\omega_a^2 \sin^2\theta\right)\hat{\mathbf{r}} - r\,\omega_a^2 \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} + 2r\,\omega_a\,\omega_b \cos\theta\,\hat{\boldsymbol{\phi}}$$

The North Pole of the pink ring corresponds to $\theta = 0$ in the spherical polar coordinate system (polar angles are measured from the north pole down) so the acceleration is zero since $\sin 0 = 0$.

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3. Laplace' s equation in spherical coordinates is :

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)$$

Since our function :

$$V = r^n \cos\theta$$

has no ϕ dependence, we know that $\partial V/\partial\phi$ is zero, and we can omit the last term in Laplace' s equation. Substituting our function into Laplace' s equation gives :

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(n\,r^{n-1}\cos\theta\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(-r^n\sin\theta\right)\right) = 0$$

In the first term on the right, $\cos\theta$ is a constant as is r in the second term, allowing us to write :

$$\nabla^2 V = \frac{n\cos\theta}{r^2}\frac{d}{dr}\left(r^{n+1}\right) - \frac{r^{n-2}}{\sin\theta}\frac{d}{d\theta}\left(\sin^2\theta\right) = 0$$

Doing the indicated differentiations :

$$\nabla^2 V = n(n+1)\cos\theta\, r^{n-2} - \frac{2r^{n-2}\sin\theta\cos\theta}{\sin\theta} = 0$$

Factoring out common terms :

$$\nabla^2 V = r^{n-2}\cos\theta\,(n(n+1)-2) = 0 \Rightarrow n = -2,\ 1$$

For each of the differential equations, we will assume solutions of the form :

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with derivatives :

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \qquad\qquad y'' = \sum_{n=2}^{\infty} n\,(n-1)\, a_n x^{n-2}$$

In each of the solutions below, I will be explicit about lower limits, but will often omit upper limits on the summations since they are infinity in all cases.

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4. We know the solutions to this equation are sin and cos, let' s derive them using series techniques :

$$y'' + y = 0 \Rightarrow \sum_{n=2}^{\infty} n\,(n-1)\, a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Re - index the first summation by setting k = n - 2 and obtain :

$$\sum_{n=0} (n+2)\,(n+1)\, a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining into one summation :

$$\sum_{n=0}^{\square} x^n[(n+2)\,(n+1)\, a_{n+2} + a_n] = 0 \Rightarrow a_{n+2} = \frac{-a_n}{(n+2)\,(n+1)}$$

This recursion relation shows there are two branches of the solution, and we obtain for coefficients :

$$a_2 = \frac{-a_0}{2} \qquad a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2} \qquad a_6 = \frac{-a_4}{6.5} = \frac{-a_0}{6!}$$

$$a_3 = \frac{-a_1}{3\cdot 2} \qquad a_5 = \frac{-a_3}{5.4} = \frac{a_1}{5!} \qquad a_7 = \frac{-a_5}{7\cdot 6} = -\frac{a_0}{7!}$$

And our series solution is :

$$y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

We can easily see that the first term on the right is simply cos x, and the second is sin x.

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5. $y'' + x\,y = 0$

$$\sum_{n=2}^{\infty} n\,(n-1)\, a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

Multiply through by the x in the second term and get :

$$\sum_{n=2}^{\infty} n\,(n-1)\, a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Re - index by setting n → n - 2 in the first sum and n → n - 1 in the second sum to get :

$$\sum_{n=0} (n+2)\,(n+1)\, a_{n+2} x^n + \sum_{n=1} a_{n-1} x^n = 0$$

We strip out the first term in the first sum so that both sums have the same lower limits :

$$2\,a_2 \;+\; \sum_{n=1} [(n+2)\,(n+1)\,a_{n+2} + a_{n-1}]\,x^n \;=0$$

Yields the recursion relation :

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)\,(n+1)}$$

and also the result :

$$a_2 = 0$$

This recursion relation looks similar to the one obtained in problem 4, except notice now that the n coefficient is a multiple of the (n - 3) coefficient. Since we know that a_2 =0, we also know that a_2 = a_5 = a_8 = ... = 0

The recursion relation yields :

$$a_3 \;=\; \frac{-\,a_0}{3\cdot 2} \qquad a_6 = \frac{-a_3}{6\cdot 5} \;=\frac{a_0}{6\cdot 5\cdot 3\cdot 2}$$

$$a_4 \;= \frac{-a_1}{4\cdot 3} \qquad a_7 \;= \frac{-a_4}{7\cdot 6} \;=\frac{a_0}{7\cdot 6\cdot 4\cdot 3}$$

The solution is :

$$y = a_0\left(1-\frac{x^3}{3\cdot 2}+\frac{x^6}{6\cdot 5}-...\right)+a_1\left(x\;-\;\frac{x^4}{4\cdot 3}+\frac{x^7}{7\cdot 6\cdot 4\cdot 3}+...\right)$$

These solutions, which generate functions which oscillate in some regions (like trig functions) are called Airy Functions and appear in the study of optics (diffraction).

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6. $y'' - 2\,x\,y' - 2\,y \;=\; 0$

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} \;-\; 2\,x \sum_{n=1}^{\infty} n\,a_n\,x^{n-1} \quad -2\sum_{n=0}^{\infty} a_n\,x^n \;=\; 0$$

After multiplying through by x in the second summation :

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} \;-\; 2\sum_{n=1}^{\infty} n\,a_n\,x^{n} \quad -2\sum_{n=0}^{\infty} a_n\,x^n \;=\; 0$$

We need only re - index in the first sum by setting n → n - 2 :

$$\sum_{n=0} (n+2)\,(n+1)\,a_{n+2}\,x^n \;-\; 2\sum_{n=1}^{\infty} n\,a_n\,x^{n} \quad -2\sum_{n=0}^{\infty} a_n\,x^n \;=\; 0$$

We strip out the n = 0 terms in the first and third sums :

$$2\,a_2 - 2\,a_0 + \Sigma x^n[(n+2)\,(n+1)\,a_{n+2} - 2\,n\,a_n - 2\,a_n] \;=0$$

Since

$$2\,a_2 - 2\,a_0 \;=0 \;\Rightarrow\; a_2 = a_0$$

and the recursion relation is

$$a_{n+2} = \frac{2\,(n+1)\,a_n}{(n+2)\,(n+1)} = 2\,\frac{a_n}{n+2}$$

From which we obtain :

$$a_2 = a_0 \text{ (as expected)} \qquad a_3 = \frac{2\,a_1}{3}$$

$$a_4 = \frac{a_2}{2} = \frac{a_0}{2} \qquad a_5 = \frac{2\,a_3}{5} = \frac{4}{15}\,a_1$$

$$a_6 = \frac{a_4}{3} = \frac{a_0}{6} \qquad a_7 = 2\,\frac{a_5}{7} = \frac{8}{105}\,a_1$$

and the solution is

$$y = a_0\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots\right) + a_1\left(x + \frac{2\,x^3}{3} + \frac{4}{15}\,x^5 + \frac{8}{105}\,x^7 + \ldots\right)$$

Let' s see if we can make sense of these solutions. Notice that the even branch is just the Taylor expansion of e^{x^2}. If we solve the original differential equation using the DSolve routine we obtain:

```
DSolve[y''[x] - 2 x y'[x] - 2 y[x] == 0, y[x], x]
```

$$\left\{\left\{y[x] \to e^{x^2}\,C[2] + \frac{1}{2}\,e^{x^2}\,\sqrt{\pi}\,C[1]\,\text{Erf}[x]\right\}\right\}$$

and see that in fact one solution is merely e^{x^2}, and the other is a multiple of this solution involving the “error function” which is defined as:

$$\text{erf}\,(x) = \frac{2}{\pi}\int_0^x e^{-t^2}\,dt$$

It is the area under a Gaussian curve from the origin to x. You might have learned in multivariable the "trick" that allows you to compute the integral of this function from 0 to ∞ (or from - ∞ to ∞), but the integral from 0 to any finite value of x must be computed numerically. This function arises frequently in statistics and in solving heat diffusion problems. If we use the Mathematica series function, we can verify that our second solution is accurate :

```
Sqrt[π]/2 Series[Erf[x] Exp[x^2], {x, 0, 8}]
```

$$x + \frac{2\,x^3}{3} + \frac{4\,x^5}{15} + \frac{8\,x^7}{105} + O[x]^9$$

Which matches our series solution.

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7. $y'' - x^2\,y' - y = 0$

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - x^2 \sum_{n=1}^{\infty} n\,a_n\,x^{n-1} - \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Multiplying by x^2 in the second sum and re - indexing :

$$\sum_{n=0} (n+2)\,(n+1)\,a_{n+2}\,x^n - \sum_{n=2}^{\infty} (n-1)\,a_{n-1}\,x^n - \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Stripping out the n = 0 and n = 1 terms in the first and third sums :

$$2\,a_2 + 6\,a_3\,x - a_0 - a_1\,x + \Sigma x^n [(n+2)\,(n+1)\,a_{n+2} - (n-1)\,a_{n-1} - a_n] = 0$$

We obtain the recursion relation :

$$a_{n+2} = \frac{(n-1)\,a_{n-1} + a_n}{(n+2)\,(n+1)}$$

and the "stripped out" terms yield :

$$2\,a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$6\,a_3\,x - a_1\,x = 0 \Rightarrow a_3 = \frac{a_1}{6}$$

$$n = 1 \Rightarrow a_3 = \frac{a_1}{6} \text{ (as expected)}$$

$$n = 2 \Rightarrow a_4 = \frac{a_1 + a_2}{12} = \frac{a_1}{12} + \frac{a_0}{24}$$

$$n = 3 \Rightarrow a_5 = \frac{2\,a_2 + a_3}{20} = \frac{a_2}{10} + \frac{a_3}{20} = \frac{a_0}{20} + \frac{a_1}{120}$$

and the series soluton is

$$y = a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^5}{20} + \ldots\right) + a_1 \left(x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{120} + \ldots\right)$$

We can test our solution against a numerical solution of the differential equation. Using the NDSolve program, we can determine the numerical solution to our ODE via :

```
In[46]:= s = NDSolve[{y''[x] - x^2 y'[x] - y[x] == 0, y[0] == 1, y'[0] == 1}, y, {x, 0, 2}]

Out[46]= {{y → InterpolatingFunction[{{0., 2.}}, <>]}}
```

Where I use as initial condition y (0) = y' (0) = 1. In our series solution, this translates as

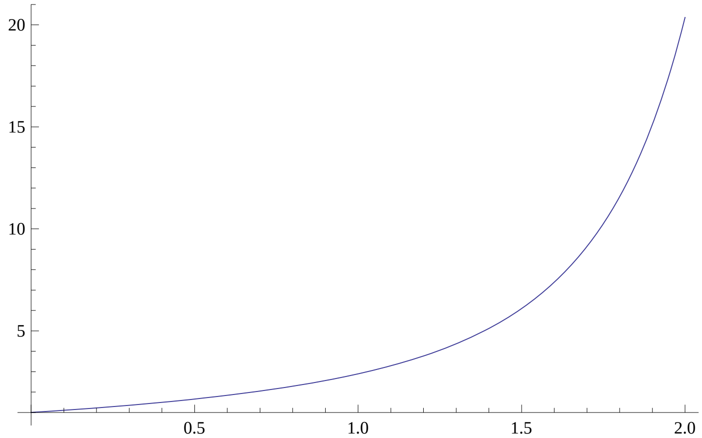
$$a_0 = a_1 = 1$$

values you can verify by setting x = 0 in the function and the first derivative.

We can plot our solution :

In[47]:=
```
Plot[Evaluate[y[x] /. s], {x, 0, 2}, PlotRange → All]
```

Out[47]=



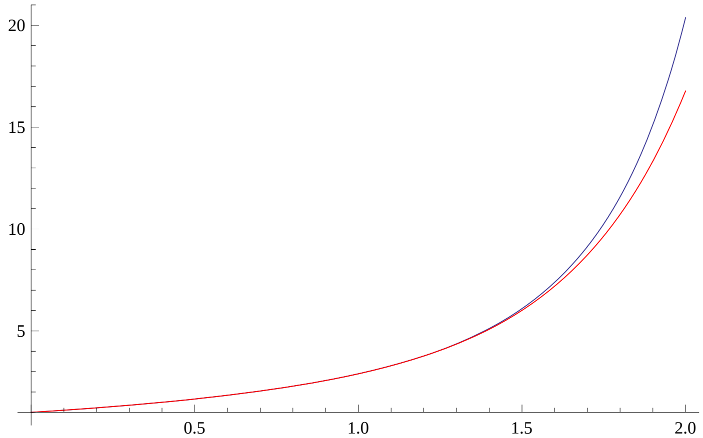
and we see the value of y rises rapidly with x, so we will need to use quite a few terms in our series solution to match it. Below I write a short program to compute a coefficients and thus the series solution, and plot it with respect to the numerical solution :

In[48]:=
```
Clear[f, a]
a[0] = 1; a[1] = 1; a[2] = a[0] / 2;
a[n_] := ((n - 3) a[n - 3] + a[n - 2]) / (n (n - 1))
f = Sum[a[n] x^n, {n, 0, 10}];
s = NDSolve[{y''[x] - x^2 y'[x] - y[x] == 0, y[0] == 1, y'[0] == 1}, y, {x, 0, 2}]
```

Out[52]=
```
{{y → InterpolatingFunction[{{0., 2.}}, <>]}}
```

In[53]:=
```
g1 = Plot[Evaluate[y[x] /. s], {x, 0, 2}, PlotRange → All];
g2 = Plot[f, {x, 0, 2}, PlotStyle → Red];
Show[g1, g2]
```

Out[55]=



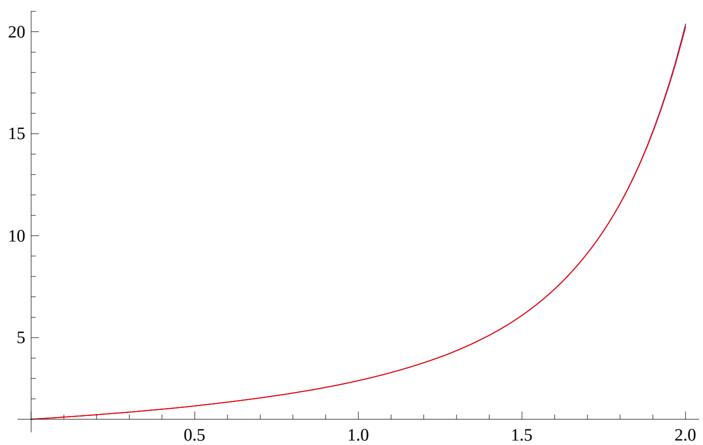
And we see that we have decent agreement to about x = 1.8 with some variation after that. If we use more terms in the series :

In[56]:=
```
Clear[f, a]
a[0] = 1; a[1] = 1; a[2] = a[0] / 2;
a[n_] := ((n - 3) a[n - 3] + a[n - 2]) / (n (n - 1))
f = Sum[a[n] x^n, {n, 0, 20}];
s = NDSolve[{y''[x] - x^2 y'[x] - y[x] == 0, y[0] == 1, y'[0] == 1}, y, {x, 0, 2}]
```

Out[60]=
```
{{y → InterpolatingFunction[{{0., 2.}}, <>]}}
```

In[61]:=
```
g1 = Plot[Evaluate[y[x] /. s], {x, 0, 2}, PlotRange → All];
g2 = Plot[f, {x, 0, 2}, PlotStyle → Red];
Show[g1, g2]
```

Out[63]=



you can see that we achieve better agreement, suggesting that our series solution is valid.