PHYS 301
HOMEWORK #12

Solutions

1. Refer to the accompanying figure (in another link). Use Legendre polynomials to express the potential due to the indicated electric quadrupole.

a) ***Solution*** : We apply the principle of superposition to write the potential as the sum of the individual potentials :

$$V = \frac{-kq}{r_1} - \frac{kq}{r_2} + 2\frac{kq}{r} = kq\left(\frac{2}{r} - \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right)$$

We will follow the procedure we used in solving the dipole problem and use the law of cosines to write r_1 and r_2 in terms of r, a and θ :

$$r_1 = r\sqrt{1 + (a/r)^2 - 2(a/r)\cos\theta}$$

$$r_2 = r\sqrt{1 + (a/r)^2 + 2(a/r)\cos\theta}$$

Using these expressions in the equation for potential allows us (again, following the example of the dipole) to write the potential in terms of Legendre series :

$$V = kq\left(\frac{2}{r} - \frac{1}{r}\left(\sum_{m=0}^{\infty} P_m(\cos\theta)(a/r)^m + \sum_{m=0}^{\infty}(-1)^m P_m(\cos\theta)(a/r)^m\right)\right)$$

The summations here look very similar to these we encountered in the dipole case with one significant difference. Here, since the charges at ± a have the same sign, we add the two sums (remember, in the dipole case the charges were of different signs and so we subtracted the sums). In the case of the quadrupole, all odd m terms cancel, leaving us only with even terms :

$$V = kq\left(\frac{2}{r} - \frac{2}{r}\sum_{m=0,\text{even}}^{\infty} P_m(\cos\theta)(a/r)^m\right) = \frac{2kq}{r}\left(1 - \sum_{m=0,\text{even}}^{\infty} P_m(\cos\theta)(a/r)^m\right)$$

It would be perfectly acceptable to leave the answer in this form. However, we can go a bit deeper by noting that the m = 0 term in the sum is simply 1, so that the lead term in the expansion of the potential is the m = 2 term :

$$V = -\frac{2kq}{r}\left(P_2(\cos\theta)(a/r)^2 + P_4(\cos\theta)(a/r)^4 + P_6(\cos\theta)(a/r)^6 + \ldots\right)$$

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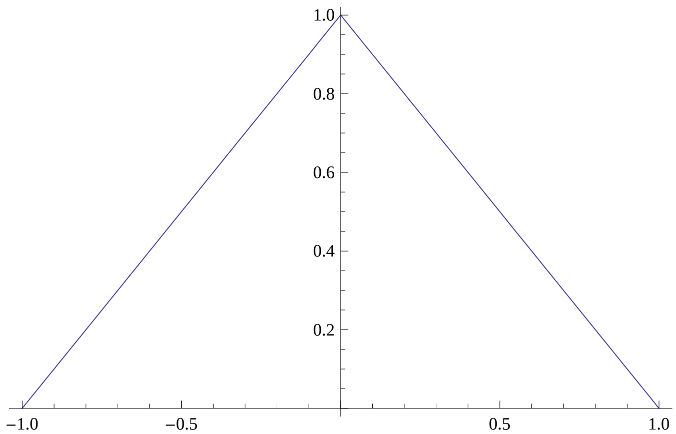
2. Expand the following as Legendre series (you may verify results using Mathematica but must show all your work by hand) :

a) $x^2 - x$

b) $7x^4 - 3x + 1$

c) f (x) =



(5 pts for each series)

Solutions : For all series, we will use the basic definitions :

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(x)$$

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

a) We compute the coefficients via :

$$c_0 = \frac{1}{2} \int_{-1}^{1} \left(x^2 - x\right) \cdot 1\, dx = \frac{1}{3}$$

$$c_1 = \frac{3}{2} \int_{-1}^{1} \left(x^2 - x\right) \cdot x\, dx = -1$$

$$c_2 = \frac{5}{2} \int_{-1}^{1} \left(x^2 - x\right) \cdot \frac{1}{2}\left(3x^2 - 1\right) dx = \frac{5}{4} \int_{-1}^{1} 3x^4 - 3x^3 - x^2 + x\, dx = \frac{2}{3}$$

c_3 and all higher order coefficients are zero (as you might expect from a quadratic function).

The Legendre series is :

$$f(x) = \frac{1}{3} P_0(x) - P_1(x) + \frac{2}{3} P_2(x) = \frac{1}{3} - x + \frac{2}{3} \cdot \frac{1}{2}\left(3x^2 - 1\right) = x^2 - x$$

Alternately, we could have computed this series using only algebra :

$$f(x) = c_0 + c_1 P_1 + c_2 P_2 = c_0 + c_1 x + c_2 \cdot \frac{1}{2}\left(3x^2 - 1\right) = x^2 - x$$

Grouping by power of x and equating to our function :

$$\left(c_0 - \frac{1}{2} c_2\right) + c_1 x + \frac{3}{2} c_2 x^2 = x^2 - x$$

Equating coefficients of like powers of x yields :

$c_0 - \frac{1}{2} c_2 = 0 \Rightarrow c_0 = \frac{c_2}{2}$

$c_1 = -1$

$c_2 = \frac{2}{3} \Rightarrow c_0 = \frac{1}{3}$

b) $c_0 = \frac{1}{2}\int_{-1}^{1} 7x^4 - 3x + 1\, dx = \frac{12}{5}$

$c_1 = \frac{3}{2}\int_{-1}^{1}\left(7x^4 - 3x + 1\right)\cdot x\, dx = -3$

$c_2 = \frac{5}{2}\int_{-1}^{1}\left(7x^4 - 3x + 1\right)\left(\frac{1}{2}\cdot\left(3x^2 - 1\right)\right) dx = \frac{5}{4}\int_{-1}^{1}\left(21x^6 - 7x^4 - 9x^3 + 3x^2 + 3x - 1\right) dx = 4$

A similar analysis will yield that $c_3 = 0$ and $c_4 = 8/5$, yielding the Legendre series :

$$f(x) = \frac{12}{5} P_0 - 3P_1 + 4P_2 + \frac{8}{5} P_4$$

c) We could compute coefficients as we have done in the first two examples, however, we can make our lives a little simpler by looking at the graph of the function and realizing that this is an even function. Since Legendre polynomials of odd order are odd (and Legendre polynomilals of even order are even), we can determine immediately from symmetry arguments that all odd coefficients will be zero (since the integrand is the product of an even function and an odd polynomial). The coefficients for the even functions can be written as :

$c_{even} = 2\cdot\frac{(2m+1)}{2}\int_0^1 (1 - x)\, P_m(x)\, dx$

Computing even coefficients :

$c_0 = \int_0^1 (1 - x)\, dx = \frac{1}{2}$

$c_2 = 5\int_0^1 (1 - x)\cdot\frac{1}{2}\left(3x^2 - 1\right) dx = \frac{-5}{8}$

$$c_4 = 9\int_0^1 (1-x)\cdot\frac{1}{8}\left(3-30\,x^2+35\,x^4\right)dx =$$

$$\frac{9}{8}\int_0^1\left(-35\,x^5+35\,x^4+30\,x^3-30\,x^2-3\,x+3\right)dx = \frac{3}{16}$$

$$f\,(x) = \frac{1}{2}\,P_0-\frac{5}{8}\,P_2+\frac{3}{16}\,P_4$$

And let' s check our results :

In[1]:=
```
Clear[c, x, f, m]
f[x_] := 1 - Abs[x]
c[m_] := c[m] = ((2 m + 1 ) / 2) Integrate[f[x] LegendreP[m, x], {x, -1, 1}]
Do[Print["c[", m, "] = ", c[m]], {m, 0, 10}]
g1 = Plot[Sum[c[m] LegendreP[m, z], {m, 0, 14}], {z, -1, 1}, PlotStyle → {Dashed, Magenta}];
g2 = Plot[f[x], {x, -1, 1}];
Show[g1, g2]
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```
c[0] = 1/2
c[1] = 0
c[2] = -5/8
c[3] = 0
c[4] = 3/16
c[5] = 0
c[6] = -13/128
c[7] = 0
c[8] = 17/256
c[9] = 0
c[10] = -49/1024
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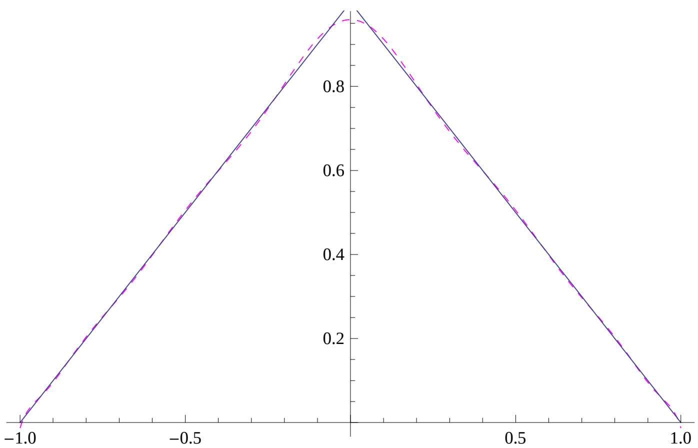
Out[7]=



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3. The generating function for Bessel' s functions of the first kind (solutions to the Bessel differential equation, see sections 12 - 15 in Boas) is :

$$g(x, t) = e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

where J_n is the n^{th} order Bessel function of the first kind. Use the generating function to show that:

a) $J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$

Solution : We begin by taking partial derivatives of both sides of the generating function equation with respect to t :

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial t} e^{(x/2)(t-1/t)} = (x/2)\left(1+\frac{1}{t^2}\right) e^{(x/2)(t-1/t)} = \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1} \tag{1}$$

Now, recalling that :

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n\,, \tag{2}$$

we can rewrite the middle term in eq. (1) as :

$$(x/2)\left(1+\frac{1}{t^2}\right) e^{(x/2)(t-1/t)} =$$

$$(x/2)\left(1+\frac{1}{t^2}\right) \sum_{n=-\infty}^{\infty} J_n(x)\, t^n = (x/2)\left\{\sum_{n=-\infty}^{\infty} J_n(x)\, t^n + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-2}\right\} \tag{3}$$

$$(x/2)\left\{\sum_{n=-\infty}^{\infty} J_n(x)\, t^n + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-2}\right\} = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1} \tag{4}$$

Now, we know from our previous work in series solutions that we wish to equate the coefficients of like powers of t. Now, for specificity, say we want to equate the coefficients of the t^3terms in each summation. In order to select the t^3 terms, the value of n must equal to 4 in the last summation on the right; n= 3 in the first summation on the left, and n must be 5 in the second summation on the left. If we generalize this result to any exponent, we see that our sums will satisfy:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

b) $J_{n-1}(x) - J_{n+1}(x) = 2\,\dfrac{d J_n(x)}{dx}$

Solution : Since we are asked to find an expression for the derivative of the Bessel function with respect to x, we might surmise that we will differentiate the generating function with respect to x :

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} e^{x/2\,(t-1/t)} = \frac{1}{2}\left(t-\frac{1}{t}\right) e^{x/2\,(t-1/t)} = \frac{\partial}{\partial x} \sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

Now, we recognize the exponential term is just the generating function for Bessel functions, so we can rewrite this expression as :

$$\frac{1}{2}\left(t-\frac{1}{t}\right) e^{x/2\,(t-1/t)} =$$

$$\frac{1}{2}\left(t-\frac{1}{t}\right) \sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \frac{1}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n+1} - \frac{1}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-1} = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

Thus, if we wish to equate terms of the nth power of t, we set our coefficient equal to n in the final summation, and therefore equal to n - 1 coefficient in the first term on the left and the n + 1 coefficient for the second term on the left, giving us :

$$\frac{1}{2}\, J_{n-1}(x) - \frac{1}{2}\, J_{n+1}(x) = J_n{}'(x)$$

(10 pts each part)