PHYS 301

HOMEWORK #13-- SOLUTIONS

1. The wave equation is :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Since we have that u = f (x - vt) and u = f (x + vt), we substitute these expressions into the wave equation.Starting with u = f (x - vt) :

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial f}\frac{\partial f}{\partial x} = f'(x - vt) * 1 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = f''(x - vt)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial f}\frac{\partial f}{\partial t} = f'(x - vt)(-v) \text{ and } \frac{\partial^2 u}{\partial t^2} = f''(x - vt)(-v)^2 = v^2 f''(x - vt)$$

Using these results we get :

$$f''(x - vt) = \frac{1}{v^2}\left(v^2 f''(x - vt)\right) \Rightarrow f''(x - vt) = f''(x - vt)$$

which shows that u = f (x - vt) satisfies the wave equation. An identical analysis will show that f (x + vt) satisfies the wave equation.

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2. #1/626 : This problem starts out very similar to the problems done in class and in the text. Given that we are solving Laplace' s equation in Cartesian coordinates, we know the general solution will be :

$$T(x, y) = (A\cos kx + B\sin kx)\left(C e^{ky} + D e^{-ky}\right)$$

Applying boundary conditions, we can set C = 0 since T must go to zero as y gets large; A = 0 since T is zero at x = 0, and the requirement that T = 0 when x = 10 cm requires that k = n π /10. The general equation then becomes :

$$T(x, y) = \Sigma B_n \sin(n\pi x/10)\, e^{-(n\pi y/10)}$$

The boundary condition at the lower edge yields :

$$T(x, 0) = \Sigma B_n \sin(n\pi x/10) = x$$

We recognize this as the Fourier sine series for f (x) = x. Remember that we have to extend f (x) = x so that we have an odd function that is 2 L periodic on (-10, 10). Remember also that since we want the sin series to match the boundary condition, we need the extended function to be odd on (-10, 10). With these taken into account, we know that the B_n coefficients are the coefficients of the Fourier sine series, so we obtain :

$$B_n = b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx = \frac{2}{10}\int_0^{10} x\sin(n\pi x/10)\,dx$$

Evaluating the integral :

In[311]:=
```
Clear[x, n]
(2 / 10) Integrate[x Sin[n π x / 10], {x, 0, 10}]
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Out[312]= $-\frac{20\,(n\,\pi\,\text{Cos}[n\,\pi] - \text{Sin}[n\,\pi])}{n^2\,\pi^2}$

For integer values of n, this gives us :

$$b_n = \frac{20\,(-1)^{n+1}}{n\,\pi}$$

(We write the exponent as n + 1 since we have a positive value for n = 1). Substituting this expression for the coefficients into the general solution, we get finally :

$$T(x, y) = \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi x/10)\,e^{-(n\pi y/10)}}{n}$$

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3. #3/626 : We have a semi - infinite plate of width π. The vertical sides are held at 0 degrees and the bottom edge has the boundary condition T (x, 0) = cos x. We are asked to find the temperature distribution throughout the plate. We know from having solved many similar problems that the general solution will be of the form :

$$T(x, y) = \Sigma B_n \sin(k x)\, e^{-k y}$$

The condition that T (π, y) = 0 implies that sin (k π) = 0 ⇒ k π = n π such that k = n , and our general solution can be written as :

$$T(x, y) = \Sigma B_n \sin(n x)\, e^{-n y}$$

The lower edge condition implies :

$$T(x, 0) = \cos x = \Sigma B_n \sin(n x)$$

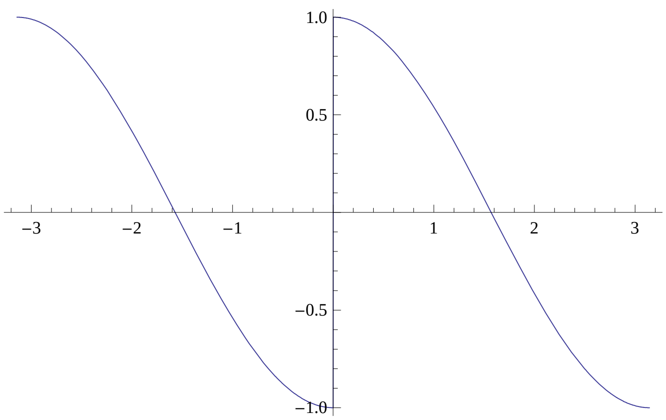
We recognize that if we can expand cos x in a Fourier sine series, we can solve for the coefficients employing the definition of the Fourier coefficients :

$$b_n = \frac{2}{\pi}\int_0^{\pi}\cos x\sin(n x)\,dx$$

Be sure to understand that we had to expand cos x as an odd function on the interval (-π, π), so that the function we are considering is :

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$$

so that on (-π, π), f (x) looks like :



In this case, the Fourier b_n coefficients are equal to the B_n coefficients we need for our general solution, so we have :

$$B_n = b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(n x)\, dx = \frac{2}{\pi}\frac{n}{n^2-1}(1+\cos(n\pi))$$

When n is odd, cos (n π) is - 1 and the coefficients are zero; for even n, cos (n π) is 1 and we have :

$$b_n = \begin{cases} \frac{4}{\pi}\frac{n}{n^2-1}, & \text{n even} \\ 0, & \text{n odd} \end{cases}$$

Finally we can use these coefficients and write our solution :

$$T(x, y) = \frac{4}{\pi}\sum_{\text{even n}}^{\infty} \frac{n}{n^2-1} \sin(n x)\, e^{-n y}$$

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4. 7/627 : This problem is similar to the second example done in the text (the rectangular plate of finite length). The process we follow will be very similar to that solution, keeping in mind that we have a different lower edge boundary condition. We know from above (and from class) that the general solution will have the form :

$$T(x, y) = (A \cos kx + B \sin kx)\left(C e^{k y} + D e^{-k y}\right)$$

Since the plate is not of infinite length, we cannot simply set C = 0, but we can match the boundary condition (BC) that T (x, 1) = 0 by setting :

$$\left(C e^{k y} + D e^{-k y}\right) = \frac{1}{2} e^{k(1-y)} - \frac{1}{2} e^{-k(1-y)} = \sinh[k(1-y)]$$

As we have seen many times, the BC that requires T (x, π) = 0 ⇒ k = n π/π = n, so that our general solution reduces to :

$$T(x, y) = \Sigma B_n \sin(n x) \sinh[n(1-y)] \tag{1}$$

The lower edge BC leads to :

$$T(x, 0) = \Sigma B_n \sin(n x) \sinh(n) = \cos x$$

Now, since we know the B are related to the coefficients of the Fourier sine series, we need to extend f (x) = cos x to make it an odd function of (-π, π). This allows us to compute :

$$B_n \sinh(n) = b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

Since L = π in this case :

$$B_n \sinh(n) = b_n = \frac{2}{\pi}\int_0^\pi \cos(x)\sin(n x)\,dx = \frac{2n(1+\cos(n\pi))}{\pi(n^2-1)} = \begin{cases} \frac{4}{\pi}\frac{n}{n^2-1}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Remember that we need to find an expression for the B for the general solution, so we note that

$$B_n = \frac{b_n}{\sinh(n)} = \frac{4}{\pi}\frac{n}{(n^2-1)\sinh(n)} \text{ for n even}$$

Substituting this expression for B into eq, (1) above gives :

$$T(x, y) = \frac{4}{\pi}\sum_{\text{even}}^{\infty}\frac{n\sin(n x)\sinh[n(1-y)]}{\sinh(n)(n^2-1)}$$

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5. 2/632 : This problem is similar to the first example in the book (p.629) and first example of the heat diffusion equation done in class.For a one dimensional bar, we expect a general solution of the form :

$$u(x, t) = \Sigma B_n \sin(k x)\, e^{-k^2\alpha^2 t}$$

For all times t > 0, we are told that u (0, t) = 0 and u (0, 10) = 0. The latter condition tells us that sin (10 k) = 0 ⇒ k = n π/10. The initial boundary condition (for t ≤ 0) is u (x, 0) = 100, so that incorporating these two results into our general solution gives us :

$$u(x, 0) = \Sigma B_n \sin(n\pi x/10) = 100$$

We recognize immediately that the B_n are the Fourier b_n coefficients when f (x) = 100 and L = 10, so that we compute :

$$B_n = b_n = \frac{2}{10}\int_0^{10} 100\sin(n\pi x/10)\,dx = -\frac{200(-1+\cos(n\pi))}{n\pi} = \begin{cases} 400/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

and the solution is :

$$u(x, t) = \frac{400}{\pi}\sum_{\text{odd } n}^{\infty}\frac{\sin(n\pi x/10)\, e^{-(n\pi\alpha/10)^2 t}}{n}$$

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6. 632/5 : . This problem differs from the previous one in that the final steady - state configuration produces temperatures different from zero.Since the sum of solutions is also a solution to the general case, we add the final result and our general solution is of the form :

$$u(x, t) = \Sigma(a_n\cos kx + b_n\sin kx)\, e^{-k^2\alpha^2 t} + u_f = \Sigma(a_n\cos kx + b_n\sin kx)\, e^{-k^2\alpha^2 t} + 100$$

where u_f represents the temperature distribution as t grows very large. Given that the two outer faces are held at 100^o, we expect that as t grows large, the final temperature distribution simply becomes u_f = 100. We have not yet discarded either the sin or cos solution for the spatial component of the solution; we must use boundary conditions to determine which function (i.e., sin or cos) should be retained in the solution.

If the temperature at x=0 must be 100^o for all times t > 0, then the summation must equal zero in order for u(0,t) = 100. Since cos 0 ≠0, only the sin solution works, and our solution will involve only sin terms. In order for the BC sin(10 k) = 0, we know from many previous examples that k = n π/2 (the total width of the solid = 2), and we have:

$$u(x, t) = \Sigma B_n \sin(n \pi x / 2) e^{-(n \pi \alpha/2)^2 t} + 100 \qquad (2)$$

Applying the condition that at t = 0 the temperature distribution is :

$$u(x, 0) = \begin{cases} 100\,x, & 0 < x < 1 \\ 100\,(2 - x), & 1 < x < 2 \end{cases}$$

Then, we can write :

$$u(x, 0) = \Sigma B_n \sin(n \pi x / 2) + 100$$

$$u(x, 0) - 100 = \Sigma B_n \sin(n \pi x / 2)$$

and we can see that the B_n coefficients are simply the Fourier coefficients for the function u (x, 0) - 100 on the interval (0, 2). We find these coefficients from :

$$B_n = b_n = \frac{2}{2}\left[\int_0^1 (100\,x - 100) \sin(n \pi / 2)\, dx + \int_1^2 -100\,(x - 1)\, \text{si}\,(n \pi / 2)\, dx\right]$$

When evaluated, these coefficients are :

$$B_n = b_n = \begin{cases} 0, & \text{n even} \\ \frac{8}{n^2 \pi^2} - \frac{4}{n \pi}, & n = 1 \text{ Mod } 4 \,(n = 1, 5, 9, \ldots) \\ \frac{-8}{n^2 \pi^2} - \frac{4}{n \pi}, & n = 3 \text{ Mod } 4 \,(n = 3, 7, 11, \ldots) \end{cases}$$

Substitute these values for B_n into equation (2), and we have a complete solution for the problem.

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7. 637/2 : This is a standard wave equation for a string with zero initial velocity and whose ends are fixed such that y (0, t) = y (L, t) = 0.

Solving the wave equations yields a general solution of the form :

$$y(x, t) = \Sigma (a_k \cos kx + b_k \sin kx)(c_k \cos(k v t) + d_k \sin(k v t)).$$

One spatial boundary condition (i.e., y(0,t) = 0) leads us to discard the cos kx solution since cos kx cannot be zero at x = 0. The spatial condition at x = L leads to sin (k L) = 0 ⇒ k = n π/L.

We are told that the initial velocity of the string is zero, requiring that ∂y (x, 0)/∂t = 0. This condition causes us to discard the sin (k v t) solutions, since cos (k v t), the derivative of sin (k v t),cannot be zero at t = 0. However, we can have the sin (k v t) solutions since the derivative of cos (k v t) = 0 at t = 0. Thus, our general solution has the form :

$$y(x, t) = \Sigma b_n \sin(n \pi x / L) \cos(n \pi v t / L). \tag{3}$$

As is becoming familiar, we find the values of the coefficients by applying the boundary condition at t = 0 and solving for the appropriate Fourier coefficients.The boundary condition is :

$$y(x, 0) = \begin{cases} 4 h x / L, & 0 < x < L / 4 \\ 2 h - 4 h x / L, & L / 4 < x < L / 2 \\ 0, & L / 2 < x < L \end{cases}$$

We compute the relevant Fourier coefficients by extending y (x, 0) to make it an odd function on (-L, L). Computing the Fourier sine coefficients we get :

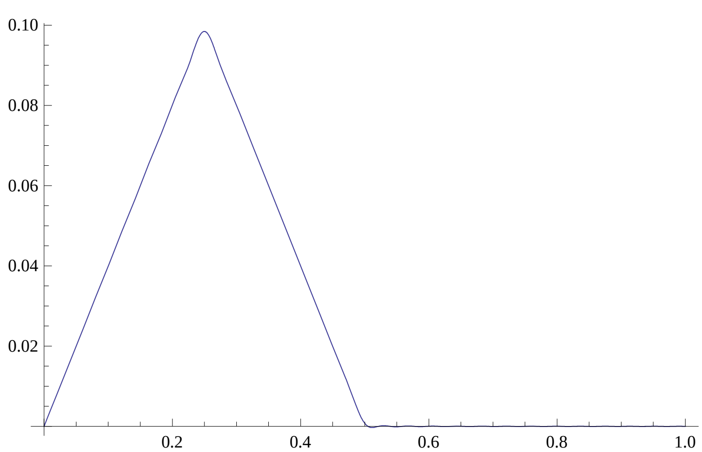
$$b_n = \frac{2}{L}\left[\int_0^{L/4} (4 h x / L) \sin(n \pi x / L)\, dx + \int_{L/4}^{L/2} (2 h - 4 h x / L) \sin(n \pi x / L)\, dx\right]$$

Using Mathematica, you can determine these coefficients to be :

$$b_n = \frac{64 h}{n^2 \pi^2} \cos(n \pi / 8) \sin^3(n \pi / 8)$$

Substituting this expression for coefficients into equation (3) will produce the general solution for the equation (Note that this problem was previously solved; see problem 24 on p. 371) .The plot below shows that these coefficients will reproduce the initial y (x, 0) condition :

```
Clear[b, x, h, L]
h = 0.1; L = 1;
b[n_] := 64 h Cos[n π / 8] Sin[n π / 8]^3 / (n π)^2
Plot[Sum[b[n] Sin[n π x / L], {n, 1, 51}], {x, 0, L}]
```



We can use the Manipulate command to enable us to simulate the motion of this wave pattern :

```
Clear[b, y, x, h, L, t, v]
h = 0.1; L = 1.0; v = 1;
b[n_] := (64 h / π^2) Cos[n π / 8] Sin[n π / 8]^3 / n^2
Manipulate[Plot[Sum[b[n] Sin[n π x / L] Cos[n π v t / L], {n, 1, 31}],
  {x, 0, L}], {t, 0, 50, 0.1}]
```

Since I have to post this as a .pdf, I cannot show the interactive nature of Manipulate; you will need to type the code into an open notebook and execute it. Note that I have to provide numerical values for h, L and v in order to allow Mathematica to produce a plot. You can play around with the values; you will find that there is a trade - off between the number of terms in the sum and the speed with which Mathematica can update the simulation. Try using the "play" option in Manipulate, slowing down the simulation until you can get a sense for how the disturbance propagates down the string.

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8. 637/5 : This is another wave equation problem with the ends fixed at y = 0 at all times. Using the previous problem as a guide, we know our solution will have the general form :

$$y(x, t) = \Sigma (A_k \cos kx + B_k \sin kx) (C_k \cos(k v t) + D_k \sin(k v t)).$$

Further, we know the cos (k x) terms will be zero because y (0, t) = 0, and we also know that k = n π x/L since y (L, t) = 0. We can conclude that our solution will look something like :

$$y(x, t) = \Sigma (B_k \sin(n \pi x / L) (C_k \cos(k v t) + C_k \sin(k v t)).$$

Now, we have to use the initial velocity profile. Unlike the previous problem, the initial velocity is not zero, this means that $\partial y(x, 0)/\partial t$ is non - zero at t = 0. This condition requires us to discard the cos (k v t) solutions. (The time derivative of cos (k v t) returns sin (k v t); sin is zero at zero so cos (k v t) cannot be part of the solution. The time derivative of sin (k v t) returns cos (k v t); since cos is non - zero at t = 0, these are the solutions we need in this case.) Now, we can apply the BC for t = 0 to obtain :

$$\frac{\partial y(x, 0)}{\partial t} = \frac{\partial}{\partial t} (\Sigma B_k \sin(n \pi x / L) \sin(n \pi v t / L))\Big|_{t=0} =$$

$$\Sigma B_k (n \pi v / L) \sin(n \pi x / L) \cos(n \pi v t / L)\Big|_{t=0} = \Sigma B_k (n \pi v / L) \sin(n \pi x / L)$$

and we are given in the text that the initial boundary condition can be expressed as :

$$f(x) = \begin{cases} 2 h x / L, & 0 < x < L / 2 \\ 2 h (1 - x / L), & L / 2 < x < L \end{cases}$$

You may remember that we enountered this function in Chapter 7 (Fourier Series), section 10, problem 23 on p. 371.

The procedure now is familiar : extend this function as an odd function on (-L, L) to ensure we obtain the Fourier sin coefficients. Recognize that the Fourier sin coeffficients are related to the cluster of coefficients in this problem via :

$$b_n = \frac{n \pi v B_n}{L} \Rightarrow B_n = \frac{L}{n \pi v} b_n$$

and compute the b_n :

$$b_n = \frac{2}{L} \int_0^L f(x) \sin(n \pi x / L)\, dx$$

Using our friend Mathematica :

```
In[327]:= Clear[h, L, x]
          (2 / L) (Integrate[2 h x / L Sin[n π x / L], {x, 0, L / 2}] +
             Integrate[Sin[n π x / L] 2 h (1 - x / L), {x, L / 2, L}])
```

Out[328]=
$$\frac{2\left(\frac{h L \left(-n \pi \mathrm{Cos}\left[\frac{n\pi}{2}\right]+2 \mathrm{Sin}\left[\frac{n\pi}{2}\right]\right)}{n^2 \pi^2} + \frac{h L \left(n \pi \mathrm{Cos}\left[\frac{n\pi}{2}\right]+2 \mathrm{Sin}\left[\frac{n\pi}{2}\right]-2 \mathrm{Sin}[n\pi]\right)}{n^2 \pi^2}\right)}{L}$$

```
In[329]:= Simplify[%, Assumptions → n ∈ Integers]
```

Out[329]=
$$\frac{8 h \mathrm{Sin}\left[\frac{n\pi}{2}\right]}{n^2 \pi^2}$$

Isn' t that a nice result? Sin (n π/2) will be zero for even values of n, and the odd values will alternate signs. Thus, we expect to obtain an alternating series consisting of only odd n terms. Computing our B_n coefficients :

$$B_n = \frac{L b_n}{n \pi v} = \frac{8 h L}{n^3 \pi^3 v}$$

Substituting these back into our solution yields :

$$y(x, t) = \Sigma B \sin(n \pi x / L) \sin(n \pi v t / L) = \frac{8 h L}{\pi^3 v} \sum_{\text{odd}}^{\infty} \frac{(-1)^{n+1} \sin(n \pi x / L) \sin(n \pi v t / L)}{n^3}$$

and this solution matches the expression given in the text.

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9. 650/8 : Hopefully by now, all the solutions to these problems are forming a pattern. Find the appropriate equation; use separation of variables to to find general solutions; apply boundary conditions to determine specific solutions. The latter step may involve writing a boundary condition as a function, and expressing that function as a Fourier or Legendre series. Here we use Legendre series. Since we are solving Laplace' s equation on a sphere, we know our solutions are of the form :

$$T(r, \theta) = \Sigma \left(A_m r^m + B_m r^{-(m+1)}\right) P_m(\cos\theta)$$

We are asked to find the solution inside the sphere, so we know the B_m coefficients are zero (otherwise the solution would diverge at r = 0). The boundary condition at r = 1 is :

$$T(1, \theta) = \begin{cases} 100, & 0 < \theta < \pi / 3 \\ 0, & \text{elsewhere} \end{cases}$$

this can be written as :

$$T(1, \theta) = 100,\ 1/2 < \cos\theta < 1 \ \text{(and 0 elsewhere)}$$

Applying the surface boundary condition to our solution gives us :

$$T(1,\ \theta) = \Sigma A_m (1)^m P_m(\cos\theta) = 100 \text{ for } 1/2 < \theta < 1$$

which makes it clear that the A_m coefficients are simply the Legendre series coefficients which we find from :

$$A_m = c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x) = \frac{2m+1}{2}\int_{1/2}^{1} 100 \cdot P_m(x)\,dx$$

The rest is easy :

$$A_0 = c_0 = \frac{1}{2}\int_{1/2}^{1} 100\,dx = \frac{1}{2}\cdot 100 \cdot \frac{1}{2} = 25$$

$$A_1 = c_1 = \frac{3}{2}\int_{1/2}^{1} 100\,x\,dx = \frac{225}{4}$$

$$A_2 = c_2 = \frac{5}{2}\int_{1/2}^{1} 100 \cdot \frac{1}{2}\left(3x^2 - 1\right)dx = \frac{375}{8}$$

$$A_3 = c_3 = \frac{7}{2}\int_{1/2}^{1} 100 \cdot \frac{1}{2}\left(5x^3 - 3x\right)dx = \frac{525}{64}$$

We can write the solution as the series :

$$T(r,\ \theta) = \Sigma A_m r^m P_m(\cos\theta) = 25\left[r^0 P_0 + \frac{9}{4} r P_1 + \frac{15}{8} r^2 P_2 + \frac{21}{64} r^3 P_3 + \ldots\right]$$

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10. 650/12 : One wrinkle in this problem is that we are given both the temperature on the surface of the sphere and on the equatorial plane. The cryptic statement at the end of the paragraph on the top of p. 650 is reminding us of Dirichlet' s conditions, in particular the result that the value of Fourier (and Legendre) series converge to the midpoint at a discontinuity. Our discontinuity occurs at the equatorial plane. Since this midpoint value on the equatorial plane is zero, we can conclude that the temperature in the upper hemisphere must be equal and opposite in sign to the temperature in the lower hemisphere. Thus (as the text instructs), we think of the surface boundary condition as a function first defined on (0, 1) and then extended as an odd function on (-1, 1). Constraining our function to be odd requires that our solution consist only of odd terms, so we only compute the coefficients for m = odd integer. We use symmetry to compute the Legendre coefficients :

$$c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\,dx = 2\cdot\frac{2m+1}{2}\int_{0}^{1} f(x) P_m(x)\,dx$$

In this case, our function on (0, 1) is $\cos^2\theta$, so setting $x = \cos\theta$ yields:

$$c_m = (2m+1)\int_{0}^{1} x^2 P_m(x)\,dx$$

$$c_1 = \frac{3}{4}$$

$c_3 = \frac{7}{24}$

$c_5 = \frac{-11}{192}$

and the interior solution becomes :

$$T(r, \theta) = \Sigma_{\text{odd m}} A_m r^m P_m(\cos\theta) = \frac{3}{4} r P_1 + \frac{7}{24} r^3 P_3 - \frac{11}{192} r^5 P_5 + ...$$