# PHYS 301
# HOMEWORK #1--Solutions

Due : Friday, 17 Jan. 2014

All assignments must be turned in at the beginning of class on the day they are due. All answers must be accompanied by complete and clear work.

1. Find the average value of $\sin^3 x$ on the interval $[0, \pi]$. Show how you solve this integral ; you may use Mathematica or integral tables to check your work, but you must explicitly show how you integrate this function).

***Solution*** : The average of a continuous function over the interval [a, b] is :

$$\overline{f(x)} = \frac{\int_a^b f(x)}{\int_a^b dx}$$

In this problem, we then have :

$$\bar{f} = \frac{\int_0^\pi \sin^3 x \, dx}{\pi}$$

We can evaluate the integral by re - writing the integrand :

$$\sin^3 x = \sin x \cdot \sin^2 x = \sin x \left(1 - \cos^2 x\right)$$

and the integral becomes :

$$\int_0^\pi \sin x \left(1 - \cos^2 x\right) dx = \int_0^\pi \sin x \, dx - \int_0^\pi \sin x \cos^2 x \, dx$$

$$= -\cos x \,|_0^\pi + \frac{1}{3} \cos^3 x \,|_0^\pi = -(-1-1) + 1/3\,(-1-1) = 4/3$$

(Compute the second integral by setting u = cos x so that du = - sin x dx). Therefore, the average value of $\sin^3 x$ in the interval $[0, \pi]$ is

$$\overline{f(x)} = \frac{\int_0^\pi \sin^3 x \, dx}{\pi} = \frac{4}{3\pi}$$

*Mathematica* interlude:

We can verify this result with Mathematica :

In[6]:=
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Integrate[Sin[x]^3, {x, 0, π}] / π
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Out[6]= $\frac{4}{3\pi}$

Additionally, we could construct a function that computes averages over a specified interval :

In[9]:=
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Clear[average, f, a, b]
average[f_, a_, b_] := Integrate[f, {x, a, b}] / (b - a)
average[Sin[x]^3, 0, π]
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Out[11]= $\frac{4}{3\pi}$

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2. Using power series as appropriate, show that :

$$e^{i x} = \cos x + i \sin x$$

where i is the imaginary number ($i = \sqrt{-1}$). You need not compute the appropriate Taylor series, you may quote the well known results.

***Solution*** : Expand the exponential in a Taylor series :

$$e^{i x} = 1 + (i x) + \frac{(i x)^2}{2!} + \frac{(i x)^3}{3!} + \frac{(i x)^4}{4!} + \frac{(i x)^5}{5!} + ...$$

And recall the properties of the imaginary number :

$$i = \sqrt{-1} \ ; \ i^2 = i \cdot i = -1; \ i^3 = i \cdot i^2 = -i; \ i^4 = i^2 \cdot i^2 = +1$$

Therefore, we can rewrite the Taylor series and separate the real and imaginary parts :

$$e^{i x} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)$$

The first series on the right is the series for cos x, and the second series is the expansion of sin x, showing that

$$e^{i x} = \cos x + i \sin x$$

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3. Consider the differential equation :

$$\left(1 - x^2\right) \frac{d^2 y}{dx^2} - 2 x \frac{dy}{dx} + k y = 0$$

Make the substitution $x = \cos\theta$ and rewrite this differential equation in terms of $y(\theta)$. (Do not attempt to solve this differential equation; we will spend considerable time investigating the properties of this equation later in the course).

***Solution*** : Our task is to transform each term in the differential equation to an expression where y is a function of $\theta$ instead of a function of x. We will go through the equation term by term, then combine those results at the end.

a) If we set x = cos $\theta$, the term in parentheses becomes :

$$1 - x^2 = 1 - \cos^2\theta = \sin^2\theta$$

b) Now we use the chain rule to transform the derivative terms :

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$

Since x = cos $\theta$, we have that dx/d$\theta$ = - sin $\theta$ or that

$$\frac{d\theta}{dx} = \frac{-1}{\sin\theta}$$

and finally that

$$\frac{dy}{dx} = \frac{-1}{\sin\theta}\frac{dy}{d\theta}$$

c) We will use this result to determine an expression for the second derivative term. It will be helpful to set

$$u = \frac{dy}{dx} = \frac{-1}{\sin\theta}\frac{dy}{d\theta}$$

which allows us to write :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{du}{dx}$$

Applying the chain rule :

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{d\theta}{dx}\frac{du}{d\theta}$$

We already know d$\theta$/dx, so we now compute du/d$\theta$ :

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{-1}{\sin\theta}\frac{dy}{d\theta}\right) = \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} - \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}$$

The second derivative term becomes finally :

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx}\frac{du}{d\theta} = \frac{-1}{\sin\theta}\left(\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} - \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)$$

d) Putting all these results together produces :

$$\sin^2\theta \cdot \frac{-1}{\sin\theta}\left(\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} - \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right) - 2\cos\theta\left(\frac{-1}{\sin\theta}\frac{dy}{d\theta}\right) + k\,y = 0$$

Simplifying :

$$\frac{-\cos\theta}{\sin\theta}\frac{dy}{d\theta} + \frac{d^2y}{d\theta^2} + 2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + ky = 0$$

Finally, combining terms yields the final form of the differential equation :

$$\frac{d^2 y}{d\theta^2} + \cot\theta \frac{dy}{d\theta} + k y = 0$$

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4. The transformation equations for Cartesian to cylindrical polar coordinates are :

$$x = \rho \cos\phi \quad \text{and } y = \rho \sin\phi$$

where $\rho$ is the distance from the origin and $\phi$ is the azimuthal angle (the angle measured counter-clockwise upward from the + x axis). Write the total derivatives for dx and dy in terms of $\rho$ and $\phi$.

***Solution*** : Consider a function of several variables :

$$f = f(x, y, z)$$

The total derivative, df, is expressed in terms of partial derivatives :

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Here, we have that x = x ($\rho$, $\phi$) and y = y ($\rho$, $\phi$). Then,

$$dx = \frac{\partial x}{\partial \rho} d\rho + \frac{\partial f}{\partial \phi} d\phi = \cos\phi\, d\rho - \rho \sin\phi\, d\phi$$

$$dy = \frac{\partial y}{\partial \rho} d\rho + \frac{\partial y}{\partial \phi} d\phi = \sin\phi\, d\rho + \rho \cos\phi\, d\phi$$

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5. Consider a particle whose distance from the origin is given by :

$$s = 3 \sin(2 t + \pi / 8) + 3 \sin(2 t - \pi / 8)$$

where s is distance and t is time. What are the amplitude and period of this function?

***Solution :*** Use the sin addition formulae :

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

Applying this to our expression for s yields :

$$s = 3 (2 \sin(2 t) \cos(\pi/8) = 6 \cos(\pi/8) \sin(2 t)$$

The amplitude is then 6 cos ($\pi$/8) = 5.54; the period is 2 $\pi$/2 = $\pi$.