PHYS 301

HOMEWORK #2-- SOLUTIONS

1. We begin with the results of homework #1 :

$$dx = \cos\phi\, d\rho - \rho \sin\phi\, d\phi$$
$$dy = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$$

Squaring and adding yields :

$$(dx)^2 + (dy)^2 = \cos^2\phi\,(d\rho)^2 - 2\,\rho\cos\phi\sin\phi\, d\rho\, d\phi +$$
$$\rho^2 \sin^2 (d\phi)^2 + \sin^2\phi\,(d\rho)^2 + 2\,\rho\cos\phi\sin\phi\, d\rho\, d\phi + \rho^2\cos^2\phi\,(d\phi)^2$$

$$= \left(\cos^2\phi + \sin^2\phi\right)(d\rho)^2 + \rho^2\left(\cos^2\phi + \sin^2\phi\right)(d\phi)^2$$
$$= (d\rho)^2 + \rho^2\,(d\phi)^2$$

This is simply the Pythagorean theorem in cylindrical polar coordinates. Notice that there is no contribution from cross terms ($d\rho\ d\phi$ terms); this is an example of an orthogonal transformation which we will study in more depth after our review of vector calculus and Einstein summation notation.

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2. Using the provided identities, we can write :

$$\cos[(m-n)\,x] + \cos[(m+n)\,x] =$$
$$(\cos(m\,x)\cos(n\,x) + \sin(m\,x)\sin(n\,x)) + ((\cos(m\,x)\cos(n\,x) - \sin(m\,x)\sin(n\,x))) =$$
$$2\cos(mx)\cos(nx) \Rightarrow \cos(m\,x)\cos(n\,x) = \frac{1}{2}\,(\cos[(m-n)\,x] + \cos[(m+n)\,x])$$

Thus,

$$\int_{-\pi}^{\pi}\cos(m\,x)\cos(n\,x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos[(m-n)\,x] + \cos[(m+n)\,x]\,dx =$$
$$\frac{1}{2}\left[\frac{\sin(m-n)\,x}{m-n}\Big|_{-\pi}^{\pi} + \frac{\sin(m+n)}{m+n}\Big|_{-\pi}^{\pi}\right] = 0$$

Since m and n are integers, all evaluations are zero since $\sin(n\,\pi) = 0$ for all integer values of n.

To evaluate :

$$\int_{-\pi}^{\pi}\sin(m\,x)\sin(n\,x)\,dx$$

use the cos addition formulae above but subract the two terms as :

$$\cos[(m-n)\,x] - \cos[(m+n)\,x] =$$
$$(\cos(m\,x)\cos(n\,x) + \sin(m\,x)\sin(n\,x)) - ((\cos(m\,x)\cos(n\,x) - \sin(m\,x)\sin(n\,x))) =$$

$$2\sin(m\,x)\sin(n\,x) \;\Rightarrow\; \cos(m\,x)\cos(n\,x) \;=\; \frac{1}{2}\,(\cos[(m-n)\,x] - \cos[(m+n)\,x])$$

Integrating yields the same results as above, showing that

$$\int_{-\pi}^{\pi} \sin(m\,x)\sin(n\,x)\,dx \;=0$$

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3. Find the Fourier coefficients and series of :

$$f(x) \;=\; \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

If we plot this function :

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Clear[f]
f[x_] := Which[-π < x < 0, -1, 0 < x < π, 1]
Plot[f[x], {x, -1, 1}]
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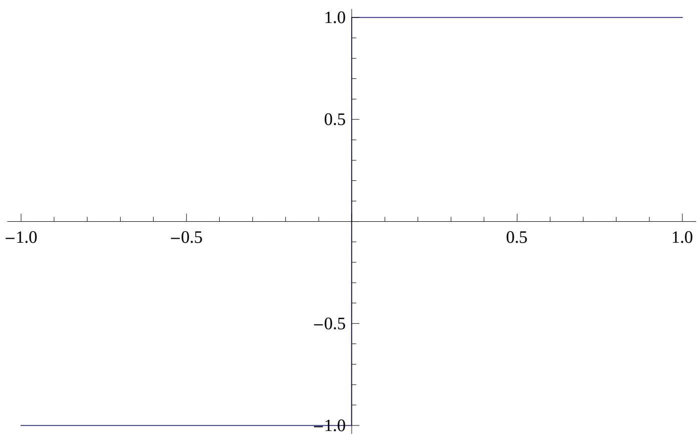

We can see easily that this is an odd function of x. Using the symmetry of this function, we can write :

$$a_0 \;=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \;=\; 0 \text{ since an the integral of an odd function is zero on an interval } [-L,\, L]$$

$$a_n \;=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,dx \;=0 \text{ since the product of an odd and even function is odd}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,dx \;=\; \frac{2}{\pi}\int_{0}^{\pi} 1\cdot\sin(n\,x)\,dx \;=\; \frac{-2}{n\,\pi}\cos(n\,x)\Big|_0^{\pi} \;=$$

$$\frac{-2}{n\,\pi}\,(\cos(n\,\pi)-1) = \frac{2}{n\,\pi}\,(1-\;(-1)^n) \;=\; \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\,\pi}, & n \text{ odd} \end{cases}$$

Thus, our Fourier series is :

$$f(x) \;=\; \frac{4}{\pi}\left[\sin x + \frac{\sin 3\,x}{3} + \frac{\sin 5\,x}{5} + \ldots\right]$$

Verifying with Mathematica, summing through the sin 31 x term and plotting three cycles of the function :

In[284]:=
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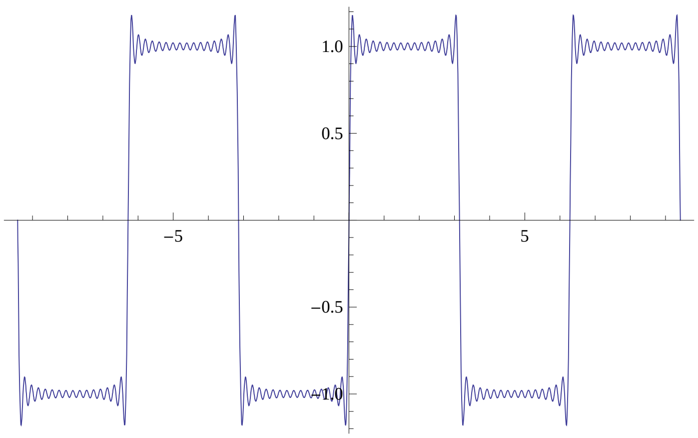
Plot[ (4 / π) Sum[Sin[n x] / n, {n, 1, 31, 2}], {x, -3 π, 3 π}]
```

Out[284]=



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4. Find the Fourier coefficients and series for :

$$f(x) = \text{Abs}[x], \; -\pi < x < \pi$$

Plotting this function shows that it is an even function :

```
Plot[Abs[x], {x, -π, π}]
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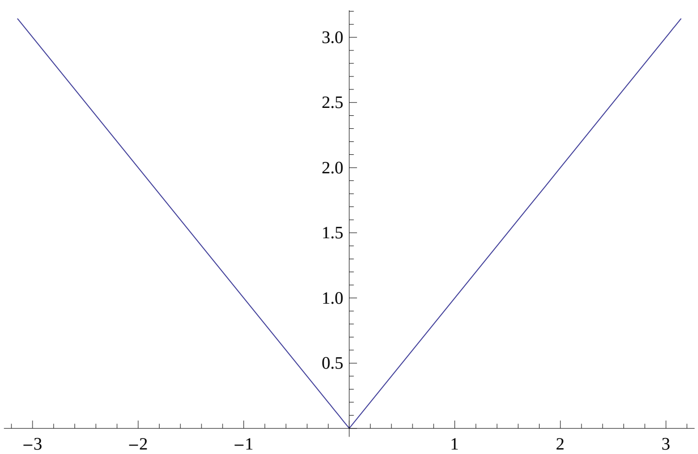


So that we can write :

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(n x) \, dx$$

$b_n = 0$ since the product of an even function (abs (x) and an odd function (sin x) is odd, and integrating an odd function over $[-L, L]$ yields zero)

Thus, we need only to use integration by parts to compute the a_n coefficients:

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(n\,x)\,dx$$

$$\text{set } u = x \Rightarrow du = dx$$

$$\text{set } dv = \cos(n\,x)\,dx \Rightarrow v = \frac{1}{n}\sin(n\,x)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(n\,x)\,dx = \frac{2}{\pi}\left[\frac{1}{n}\,x\sin(n\,x)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(n\,x)\,dx\right] =$$

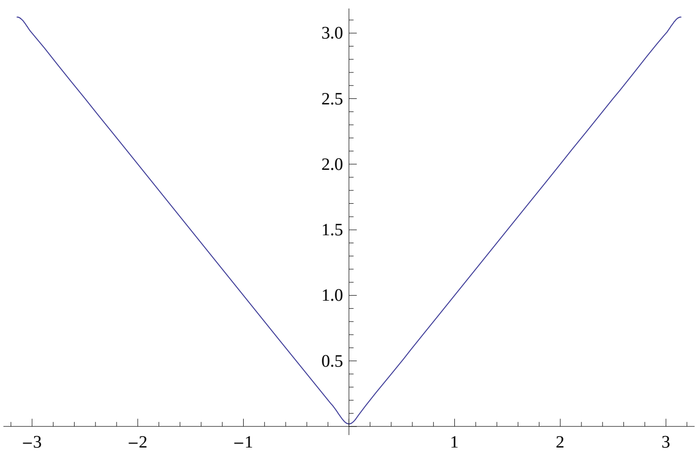
$$\frac{2}{\pi}\left[0 - \frac{1}{n^2}\left(-\cos(n\,x)\Big|_0^{\pi}\right)\right] = \frac{2}{n^2\,\pi}\left[\cos(n\,\pi) - 1\right] = \begin{cases} 0, & n \text{ even} \\ -4\big/n^2\,\pi, & n \text{ odd} \end{cases}$$

And our Fourier series is :

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\text{Cos}\,x + \frac{\text{Cos}\,3\,x}{3^2} + \frac{\cos 5\,x}{5^2} + \ldots\right]$$

Verifying via Mathematica :

```
Plot[(π / 2) - (4 / π) Sum[Cos[n x] / n^2, {n, 1, 31, 2}], {x, -π, π}]
```



5. Find Fourier coefficients and the Fourier series for :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \cos x, & 0 < x < \pi \end{cases}$$

Using the standard formulae for Fourier coefficients :

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\cos x\,dx = \frac{1}{\pi}\cdot\sin x\Big|_0^{\pi} = 0$$

We can use the results of problem 2 of this assignment to compute

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos x\cos(n\,x)\,dx$$

As long as n ≠ 1, we can write :

$$a_n = \frac{1}{\pi}\int_0^\pi \cos x \cos(n\,x)\,dx = \frac{1}{2\pi}\left[\frac{\sin(1-n)\,x}{1-n} + \frac{\sin(1+n)\,x}{1+n}\right]\Bigg|_0^\pi \qquad (1)$$

Since sin (q π) is zero for all integer values of q (and of course is zero for x = 0), this integral is zero whenever n ≠ 1. However, when n = 1, we have :

$$a_1 = \frac{1}{\pi}\int_0^\pi \cos^2 x\,dx = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

We can use the sin addition formulae to evaluate :

$$b_n = \frac{1}{\pi}\int_0^\pi \cos x \sin(n\,x)\,dx$$

$$\sin[(n+1)\,x] + \sin[(n-1)\,x] =$$
$$(\sin(n\,x)\cos x + \cos(n\,x)\sin x) + (\sin(n\,x)\cos x - \cos(n\,x)\sin x) =$$
$$2\sin(n\,x)\cos x$$

$$\text{Thus, } b_n = \frac{1}{\pi}\int_0^\pi \cos x \sin(n\,x)\,dx = \frac{1}{2\pi}\int_0^\pi \sin[(n+1)\,x] + \sin[(n-1)\,x] =$$
$$\frac{1}{2\pi}\left[-\frac{\cos(n+1)\,x}{n+1} - \frac{\cos(n-1)\,x}{n-1}\right]\Bigg|_0^\pi$$

To evaluate these terms at x = π, we use the cos addition formula and get :

$$\cos(n\pm1)\,x = (\cos n\,x \cos x \mp \sin n\,x \sin x) = \cos(n\,\pi)\cos\pi - \sin(n\,\pi)\sin\pi$$

The sin terms are zero leaving us with :

$$\cos(n\pm1)\,\pi = \cos(n\,\pi)\cos\pi = -(-1)^n$$

Using these results we can compute (remembering to evaluate at x=0 also) :

$$b_n = \frac{1}{2\pi}\left[\frac{(-1)^n+1}{n+1} + \frac{(-1)^n+1}{n-1}\right]$$

Combining fractions yields :

$$b_n = \frac{1}{2\pi}\cdot\frac{2\,n}{n^2-1}[(-1)^n+1] = \begin{cases} 0, & n \text{ odd} \\ \frac{2\,n}{\pi\left(n^2-1\right)}, & n \text{ even} \end{cases}$$

Writing the Fourier series explicitly gives :

$$f(x) = \frac{1}{2}\cos x + \frac{2}{\pi}\left[\frac{2\sin 2\,x}{2^2-1} + \frac{4\sin 4\,x}{4^2-1} + \frac{6\sin 6\,x}{6^2-1} + \ldots\right]$$

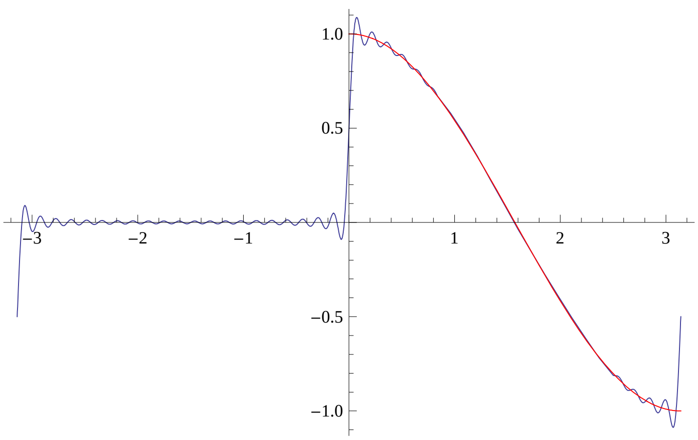
A few comments about this series expansion : 1) Many students omit the a_1 term because they look at the integral in eq. 1 and immediately conclude that $\int_0^\pi \cos x \cos(n\,x)\,dx$ is zero for all integer values of n. As you can see from the evaluation of the integral, this will be true for all values of n ***except*** when n =1. Only when n=1 will this integral yield a non-zero value, so that there will be an

a_1 term in the Fourier expansion. 2) The b_n coefficients involve a $1/(n^2-1)$ term, which may lead to concern for the convergence of the series when n = 1. However, note that the b_n terms are zero for odd values of n, so that b_1 is well defined (and equals zero).

Let' s verify that our Fourier series converges to f (x) :

In[277]:=
```
Clear[a0, a, b]
a0 = 0;
a1 = 1 / 2;
b[n_] := 2 n / (π (n^2 - 1))
g1 = Plot[a1 Cos[x] + Sum[b[n] Sin[n x], {n, 2, 42, 2}], {x, -π, π}];
g2 = Plot[Cos[x], {x, 0, π}, PlotStyle → Red];
Show[g1, g2]
```

Out[283]=



6. Find Fourier coefficients and the Fourier series for

$$f(x) = 1 - x, \quad -\pi < x < \pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-x)\,dx = \frac{1}{\pi}\cdot\left(x - \frac{x^2}{2}\right)\Big|_{-\pi}^{\pi} = 2$$

Let' s use symmetry to simplify the next integral :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-x)\cos(n\,x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{\pi}\cos(n\,x)\,dx - \int_{-\pi}^{\pi}x\cos(n\,x)\,dx\right)$$

We know that the first integrand on the right is an even function of x, and the second integrand is odd (since x is odd and cos is even, their product is odd). Therefore, we know the integral of x cos (n x) is zero, leaving us only with :

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos(n\,x)\,dx = \frac{1}{n}\sin(n\,x)\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-x)\sin(n\,x)\,dx = -\frac{2}{\pi}\int_0^{\pi}x\sin(n\,x)\,dx$$

where we again use symmetry arguments to simplify the integral. Using standard techniques of integration by parts :

$$b_n = \frac{-2}{\pi}\left[\frac{-1}{n}\ x\cos(n\,x)\Big|_0^{\pi} - \left(\frac{-1}{n}\right)\int_0^{\pi}\cos(n\,x)\,dx\right]$$

We can inspect the remaining integral and recognize it will return a sin (n x) term to be evaluated at π and 0, meaning the value of the remaining integral is zero. Thus, our final set of coefficients is :

$$b_n = \frac{2}{\pi\,n}\ x\cos(n\,x)\Big|_0^{\pi} = \frac{2}{n\,\pi}[\pi\cos(n\,\pi) - 0] = \frac{2}{n}(-1)^n$$

The first three terms of the Fourier series are :

$$f(x) = \frac{2}{2} - 2\left[\sin x - \frac{\sin 2\,x}{2} + \frac{\sin 3\,x}{3} - ...\right]$$

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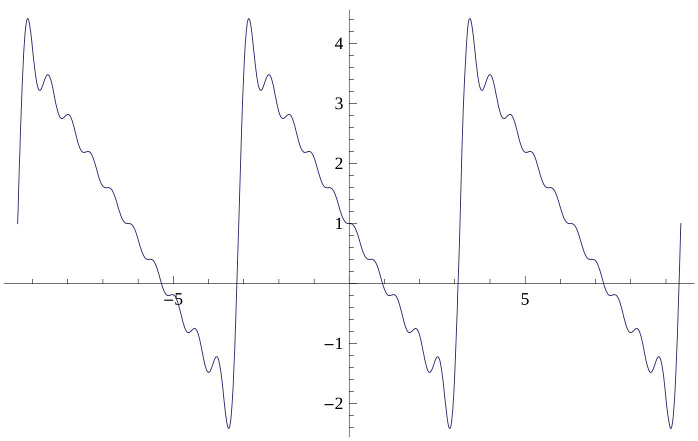
7. We will define the functions f, a and b to represent respectively the function we will plot and the Fourier coefficients. a0 is declared as a variable:

In[34]:=
```
Clear[f, a0, a, b]
a0 = 2;
a[n_] := 0;
b[n_] := 2 (-1)^n / n
f[x_] := a0 / 2 + Sum[a[n] Cos[n x] + b[n] Sin[n x], {n, 1, 10}]
Plot[f[x], {x, -3 π, 3 π}]
```

Out[39]=



To verify that this is the graph of f (x) = 1 - x on [-π, π], we superimpose the two graphs :

In[43]:=
```
g1 = Plot[f[x], {x, -3 π, 3 π}];
g2 = Plot[1 - x, {x, -π, π}, PlotStyle → Red];
Show[g1, g2]
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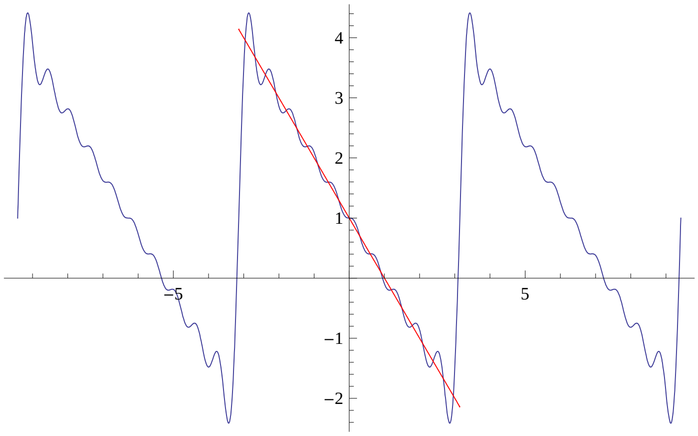
Out[45]=



In the above statements, I define the plot of the Fourier series as a variable (named ' g1') and the plot of the function over [-π, π] as the variable ' g2'. The semicolons at the end of each line suppress the output so you do not see the individual graphs. The ' Show' command combines the two graphs (and the second graph is plotted in red to make it more visible).