PHYS 301

HOMEWORK #2-- SOLUTIONS

1. We begin with the results of homework #1:

\[ dx = \cos \phi \, d\rho - \rho \sin \phi \, d\phi \]
\[ dy = \sin \phi \, d\rho + \rho \cos \phi \, d\phi \]

Squaring and adding yields:

\[ (dx)^2 + (dy)^2 = \cos^2 \phi \, (d\rho)^2 - 2 \rho \cos \phi \sin \phi \, d\rho \, d\phi + \rho^2 \sin^2 \phi \, (d\phi)^2 + \sin^2 \phi \, (d\rho)^2 + 2 \rho \cos \phi \sin \phi \, d\rho \, d\phi + \rho^2 \cos^2 \phi \, (d\phi)^2 \]

\[ = (\cos^2 \phi + \sin^2 \phi) \,(d\rho)^2 + \rho^2 (\cos^2 \phi + \sin^2 \phi) \,(d\phi)^2 \]
\[ = (d\rho)^2 + \rho^2 (d\phi)^2 \]

This is simply the Pythagorean theorem in cylindrical polar coordinates. Notice that there is no contribution from cross terms (d\rho \, d\phi terms); this is an example of an orthogonal transformation which we will study in more depth after our review of vector calculus and Einstein summation notation.

2. Using the provided identities, we can write:

\[ \cos[(m - n) \, x] + \cos[(m + n) \, x] = \]
\[ (\cos(mx) \cos(nx) + \sin(mx) \sin(nx)) + ((\cos(mx) \cos(nx) - \sin(mx) \sin(nx))) = \]
\[ 2 \cos(mx) \cos(nx) \Rightarrow \cos(mx) \cos(nx) = \frac{1}{2} (\cos[(m - n) \, x] + \cos[(m + n) \, x]) \]

Thus,

\[ \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m - n) \, x] + \cos[(m + n) \, x] \, dx = \]
\[ \frac{1}{2} \left[ \frac{\sin((m - n) \, x)}{m - n} \bigl|_{-\pi}^{\pi} + \frac{\sin((m + n) \, x)}{m + n} \bigl|_{-\pi}^{\pi} \right] = 0 \]

Since m and n are integers, all evaluations are zero since \sin(n\pi) = 0 for all integer values of n.

To evaluate:

\[ \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx \]

use the cos addition formulae above but subtract the two terms as:

\[ \cos[(m - n) \, x] - \cos[(m + n) \, x] = \]
\[ (\cos(mx) \cos(nx) + \sin(mx) \sin(nx)) - ((\cos(mx) \cos(nx) - \sin(mx) \sin(nx))) = \]
\[ 2 \sin(mx) \sin(nx) \Rightarrow \cos(mx) \cos(nx) = \frac{1}{2} (\cos((m-n)x) - \cos((m+n)x)) \]

Integrating yields the same results as above, showing that
\[ \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = 0 \]

3. Find the Fourier coefficients and series of:
\[ f(x) = \begin{cases} 
-1, & -\pi < x < 0 \\
1, & 0 < x < \pi 
\end{cases} \]

If we plot this function:

\[
\text{Clear}[f] \\
f[x_] := \text{Which}[-\pi < x < 0, -1, 0 < x < \pi, 1] \\
\text{Plot}[f[x], \{x, -1, 1\}] \\
\]

We can see easily that this is an odd function of \(x\). Using the symmetry of this function, we can write:
\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0 \text{ since an the integral of an odd function is zero on an interval } [-L, L] \\
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = 0 \text{ since the product of an odd and even function is odd} \\
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} 1 \cdot \sin(nx) \, dx = \frac{-2}{n} \cos(n \pi) \bigg|_{0}^{\pi} = \frac{-2}{n} (\cos(n \pi) - 1) = \frac{2}{n} (1 - (-1)^n) = \begin{cases} 
0, & n \text{ even} \\
\frac{4}{n \pi}, & n \text{ odd} 
\end{cases} 
\]

Thus, our Fourier series is:
\[ f(x) = \frac{4}{\pi} \left[ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right] \]
Verifying with Mathematica, summing through the sin 31 x term and plotting three cycles of the function:

\[
\text{In[284]} := \text{Plot}\left(\frac{4}{\pi} \sum_{n=1}^{31} \frac{\sin(n \pi)}{n},\{n, 1, 31, 2\},\{x, -3\pi, 3\pi\}\right)
\]

\[
\text{Out[284]} = \text{Plot}\left(\frac{4}{\pi} \sum_{n=1}^{31} \frac{\sin(n \pi)}{n},\{n, 1, 31, 2\},\{x, -3\pi, 3\pi\}\right)
\]

4. Find the Fourier coefficients and series for:

\[f(x) = \text{Abs}[x], -\pi < x < \pi\]

Plotting this function shows that it is an even function:

\[
\text{Plot}[\text{Abs}[x],\{x, -\pi, \pi\}\]
\]

So that we can write:

\[a_0 = \frac{2}{\pi} \int_{0}^{\pi} x \, dx = \pi\]

\[a_n = \frac{2}{\pi} \int_{0}^{\pi} \cos(nx) \, dx\]

\[b_n = 0\] since the product of an even function (abs(x)) and an odd function (\sin x) is odd, and integrating an odd function over \([-L, L]\) yields zero.

Thus, we need only to use integration by parts to compute the \(a_n\) coefficients:
\[ a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, dx \]

set \( u = x \Rightarrow du = dx \)

set \( dv = \cos(nx) \, dx \Rightarrow v = \frac{1}{n} \sin(nx) \)

\[
a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, dx = \frac{2}{\pi} \left[ \frac{1}{n} x \sin(nx) \right]_0^\pi - \frac{1}{n} \int_0^\pi \sin(nx) \, dx = \frac{2}{n^2 \pi} \left[ \cos(n\pi) - 1 \right] = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{n^2 \pi}, & n \text{ odd} \end{cases}
\]

And our Fourier series is:

\[ f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} \]

Verifying via Mathematica:

```mathematica
Plot[(\[Pi]/2) - (4/\[Pi]) Sum[Cos[n x]/n^2, \{n, 1, 31, 2\}], \{x, -\[Pi], \[Pi]\}]
```

5. Find Fourier coefficients and the Fourier series for:

\[ f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \cos x, & 0 < x < \pi \end{cases} \]

Using the standard formulae for Fourier coefficients:

\[ a_0 = \frac{1}{\pi} \int_0^\pi \cos x \, dx = \frac{1}{\pi} \sin x \bigg|_0^\pi = 0 \]

We can use the results of problem 2 of this assignment to compute

\[ a_n = \frac{1}{\pi} \int_0^\pi \cos x \cos(nx) \, dx \]

As long as \( n \neq 1 \), we can write:
\[
a_n = \frac{1}{\pi} \int_0^{\pi} \cos x \cos (n x) \, dx = \frac{1}{2\pi} \left[ \frac{\sin (1 - n) x}{1 - n} + \frac{\sin (1 + n) x}{1 + n} \right]_0^{\pi} \]  
\]  
(1)

Since \( \sin (q \pi) \) is zero for all integer values of \( q \) (and of course is zero for \( x = 0 \)), this integral is zero whenever \( n \neq 1 \). However, when \( n = 1 \), we have:

\[
a_1 = \frac{1}{\pi} \int_0^{\pi} \cos^2 x \, dx = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}
\]

We can use the sin addition formulae to evaluate:

\[
b_n = \frac{1}{\pi} \int_0^{\pi} \cos x \sin (n x) \, dx
\]

\[
\sin[(n + 1) x] + \sin[(n - 1) x] = (\sin (n x) \cos x + \cos (n x) \sin x) + (\sin (n x) \cos x - \cos (n x) \sin x) = 2 \sin (n x) \cos x
\]

Thus, \( b_n = \frac{1}{\pi} \int_0^{\pi} \cos x \sin (n x) \, dx = \frac{1}{2\pi} \int_0^{\pi} \sin[(n + 1) x] + \sin[(n - 1) x] = \frac{1}{2\pi} \left[ \frac{\cos (n + 1) x}{n + 1} - \frac{\cos (n - 1) x}{n - 1} \right]_0^{\pi} \]

To evaluate these terms at \( x = \pi \), we use the cos addition formula and get:

\[
\cos ((n \pm 1) \pi) = (\cos n \pi \cos x \mp \sin n \pi \sin x) = \cos (n \pi) \cos \pi \mp \sin (n \pi) \sin \pi
\]

The sin terms are zero leaving us with:

\[
\cos ((n \pm 1) \pi) = \cos (n \pi) \cos \pi = (-1)^n
\]

Using these results we can compute (remembering to evaluate at \( x=0 \) also):

\[
b_n = \frac{1}{2\pi} \left[ (-1)^n + 1 \right] \frac{(-1)^n + 1}{n + 1}
\]

Combining fractions yields:

\[
b_n = \frac{1}{2\pi} \cdot \frac{2n}{n^2 - 1} \left[ (-1)^n + 1 \right] = \begin{cases} 0, & \text{n odd} \\ \frac{2n}{\pi(2n^2 - 1)}, & \text{n even} \end{cases}
\]

Writing the Fourier series explicitly gives:

\[
f(x) = \frac{1}{2} \cos x + \frac{2}{\pi} \left[ \frac{2 \sin 2 x}{2^2 - 1} + \frac{4 \sin 4 x}{4^2 - 1} + \frac{6 \sin 6 x}{6^2 - 1} + \ldots \right]
\]

A few comments about this series expansion: 1) Many students omit the \( a_1 \) term because they look at the integral in eq. 1 and immediately conclude that \( \int_0^{\pi} \cos x \cos(n x) \, dx \) is zero for all integer values of \( n \). As you can see from the evaluation of the integral, this will be true for all values of \( n \) except when \( n=1 \). Only when \( n=1 \) will this integral yield a non-zero value, so that there will be an
1 term in the Fourier expansion. 2) The $b_n$ coefficients involve a $1/(n^2 - 1)$ term, which may lead to concern for the convergence of the series when $n = 1$. However, note that the $b_n$ terms are zero for odd values of $n$, so that $b_1$ is well defined (and equals zero).

Let's verify that our Fourier series converges to $f(x)$:

In[277]:= Clear[a0, a, b]

\[ a_0 = 0; \]
\[ a_1 = 1/2; \]
\[ b[n_] := 2 n / (\pi (n^2 - 1)) \]
\[ g1 = Plot[a1 \text{Cos}[x] + \text{Sum}[b[n] \text{Sin}[n x], \{n, 2, 42, 2\}], \{x, -\pi, \pi\}]; \]
\[ g2 = Plot[\text{Cos}[x], \{x, 0, \pi\}, \text{PlotStyle} \rightarrow \text{Red}]; \]
\[ \text{Show}[g1, g2] \]

Out[283]=

6. Find Fourier coefficients and the Fourier series for

\[ f(x) = 1 - x, \ -\pi < x < \pi \]

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - x) \, dx = \frac{1}{\pi} \left( \int_{-\pi}^{\pi} x \, dx - \int_{-\pi}^{\pi} x^2 \, dx \right) = 2 \]

Let's use symmetry to simplify the next integral:

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - x) \cos(nx) \, dx = \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \cos(nx) \, dx - \int_{-\pi}^{\pi} x \, d\cos(nx) \right) \]

We know that the first integrand on the right is an even function of $x$, and the second integrand is odd (since $x$ is odd and $\cos$ is even, their product is odd). Therefore, we know the integral of $x \cos(nx)$ is zero, leaving us only with:

\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} \cos(nx) \, dx = \frac{1}{n} \sin(nx) \bigg|_{0}^{\pi} = 0 \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - x) \sin(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) \, dx \]

where we again use symmetry arguments to simplify the integral. Using standard techniques of integration by parts:
$$b_n = \frac{-2}{\pi} \left[ \frac{-1}{n} x \cos (n \pi) \right]^\pi_0 - \left( \frac{-1}{n} \right) \int_0^\pi \cos (n x) \, dx$$

We can inspect the remaining integral and recognize it will return a \( \sin (n \pi) \) term to be evaluated at \( \pi \) and 0, meaning the value of the remaining integral is zero. Thus, our final set of coefficients is:

$$b_n = \frac{2}{n \pi} [\pi \cos (n \pi) - 0] = \frac{2}{n} (-1)^n$$

The first three terms of the Fourier series are:

$$f(x) = \frac{2}{2} - 2 \left[ \sin x - \frac{\sin 2 x}{2} + \frac{\sin 3 x}{3} - \ldots \right]$$

7. We will define the functions \( f, a \) and \( b \) to represent respectively the function we will plot and the Fourier coefficients. \( a_0 \) is declared as a variable:

```math
Clear[f, a0, a, b];

a0 = 2;
a[n_] := 0;
b[n_] := 2 (-1)^n / n

f[x_] := a0 / 2 + Sum[a[n] Cos[n x] + b[n] Sin[n x], {n, 1, 10}]

Plot[f[x], {x, -3 \[Pi], 3 \[Pi]}]
```

To verify that this is the graph of \( f(x) = 1 - x \) on \([-\pi, \pi]\), we superimpose the two graphs:
In the above statements, I define the plot of the Fourier series as a variable (named 'g1') and the plot of the function over [-π, π] as the variable 'g2'. The semicolons at the end of each line suppress the output so you do not see the individual graphs. The 'Show' command combines the two graphs (and the second graph is plotted in red to make it more visible).