PHYS 301
HOMEWORK #3--Solutions

1. Find the Fourier coefficients and Fourier series for the function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$$

You may use Mathematica to compute integrals, but you must submit your output with your homework. Then use the series you compute to show that :

$$\sum_{n=2,\text{even}}^{\infty} 2/\left(n^2-1\right) = 1$$

where the sum is over all even values of n.

Solution : We begin by computing the appropriate Fourier coefficients :

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\, dx = \frac{-1}{\pi} \cos x\Big|_0^{\pi} = \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos(n x)\, dx = \frac{1+(-1)^n}{\pi\left(1-n^2\right)} = \begin{cases} 0, & n \text{ odd} \\ \frac{-2}{\pi\left(n^2-1\right)}, & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin(n x)\, dx = \begin{cases} 0, & n \neq 1 \\ 1/2, & n = 1 \end{cases}$$

Therefore, the Fourier series can be written :

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] + \frac{1}{2}\sin x$$

Now, we know that at x = 0, f (0) = 0. If we set x = 0 in the Fourier series above we get :

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \ldots\right] \Rightarrow \frac{1}{2} = \sum_{\text{even } n}^{\infty} \frac{1}{n^2-1}$$

Verifying the Fourier series via Mathematica :

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In[13]:= Plot[(1 / π) - (2 / π) Sum[Cos[n x] / (n^2 - 1), {n, 2, 42, 2}] + Sin[x] / 2, {x, -π, π}]
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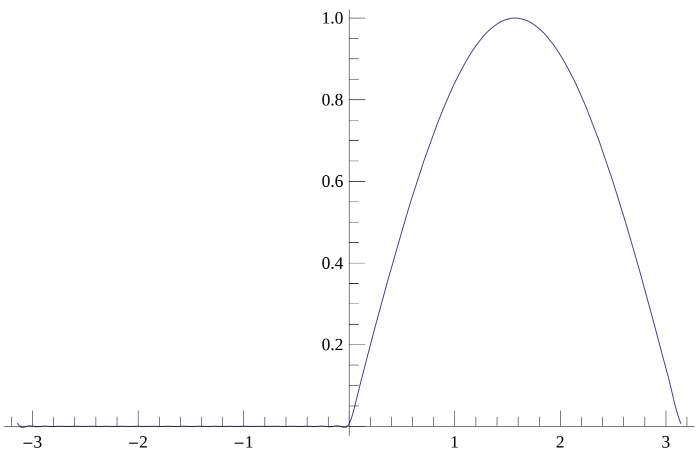

and verifying the series summation :

In[14]:=
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Sum[1 / (n^2 - 1), {n, 2, ∞, 2}]
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Out[14]= $\frac{1}{2}$

For problems 2 - 5, compute the values of the indicated expressions for integer values of n. You can end your computations when the pattern begins to repeat. (You may use *Mathematica* to verify your results, but you must do these calculations by hand.)

Solutions : In these problems, it is helpful to recall that sin and cos are 2 π periodic. This means that the pattern of coefficients will recycle when the value of n leads to an argument of sin (2 π) or cos (2 π). So in the first example, cos (n π/2) will equal cos (2 π) when n = 4. Therefore, we should expect to find four different values of 1 - cos (n π/2) before the pattern of coefficients repeats.

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2. 1 - cos (n π/2)

Solution : Cos (n π/2) is zero when n is odd; when n = 2, cos (n π) = -1 and the expression is then 1 - (-1). When n = 4, cos (2 π) = 1 and 1 - cos (2 π) = 0. We can summarize as :

n	**cos (n π / 2)**	**1 − cos (n π / 2)**
1	0	1
2	−1	2
3	0	1
4	1	0

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3. cos (n π/4)

Solution: Here, we expect to compute 8 different coefficients before the pattern repeats. We can summarize the results as:

n	**cos (n π / 4)**
1	$\sqrt{2}/2$
2	0

3	$-\sqrt{2/2}$
4	-1
5	$-\sqrt{2}/2$
6	0
7	$\sqrt{2}/2$
8	1

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4. sin (n π/4) - 1

Solution: Expecting 8 coefficients again, we have:

n	**sin (n π / 4)**	**sin (n π / 4) − 1**
1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} - 1$
2	1	0
3	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} - 1$
4	0	-1
5	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2} - 1$
6	-1	-2
7	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2} - 1$
8	0	-1

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5. sin (n π/8)

Solution: In order for sin(n π/8) = sin(2 π), the value of n = 16. Computing these 16 values:

n	**sin (n π / 8)**
1	sin (π / 8) = 0.383
2	$\frac{\sqrt{2}}{2}$
3	cos (π / 8) = 0.924
4	1
5	cos (π / 8)
6	$\frac{\sqrt{2}}{2}$
7	sin (π / 8)
8	0

n	**sin (n π / 8)**
9	−sin (π / 8)
10	$\frac{-\sqrt{2}}{2}$
11	−cos (π / 8)
12	-1
13	−cos (π / 8)
14	$\frac{-\sqrt{2}}{2}$
15	−sin (π / 8)
16	0

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6. Show that cos (i y) = cosh y and sin (i y) = i sinh y where cosh and sinh are the hyperbolic functions.

Solution: We begin by noting the definitions of sin, cos, sinh, and cosh in terms of exponential functions:

$$\cos x = \frac{e^{i x} + e^{-i x}}{2} \qquad \sin x = \frac{e^{i x} - e^{-i x}}{2 i}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

Thus, we can write cos (i y) as :

$$\cos(i y) = \frac{e^{i (i y)} + e^{-i (i y)}}{2} = \frac{e^{i^2 y} + e^{-i^2 y}}{2} = \frac{e^{-y} + e^{y}}{2} \equiv \cosh y$$

and sin (i y) becomes :

$$\sin(i y) = \frac{e^{i (i y)} - e^{-i (i y)}}{2 i} = \frac{e^{i^2 y} - e^{-i^2 y}}{2 i} = \mathbf{\frac{e^{-y} - e^{y}}{2 i} = \frac{-i (e^{-y} - e^{y})}{2}} = i \left(\frac{e^{y} - e^{-y}}{2} \right) \equiv i \sinh y$$

The boldfaced steps in the line above require recognition that $i = -1/i$, which follows from :

$$i^2 = i \cdot i = -1 \Rightarrow i = \frac{-1}{i}$$

The hyperbolic functions are the solutions to the differential equation :

$$y'' - y = 0$$

analogous to the trig functions of sin and cos as the solutions to :

$$y'' + y = 0$$

The hyperbolic functions satisfy the relationship

$$\cosh^2 x - \sinh^2 x = 1$$

reminiscent of the equation for a hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$