1. \( f(x) = \cos x, \ -1/2 < x < 1/2 \)

**Solution:** Our interval has length 1. Since our function is 2L = 1 periodic, the value of L in this case is 1/2. Since \( \cos \) is an even function, we know that all the \( b_n \) values will be zero, and we can use symmetry to simplify the calculation of \( a_0 \) and \( a_n \):

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \frac{2}{L} \int_{0}^{L} \cos x \, dx = 4 \int_{0}^{1/2} \cos x \, dx = 4 \sin x \bigg|_{0}^{1/2} = 4 \sin(1/2)
\]

Using results from earlier assignments:

\[
a_n = 4 \int_{0}^{1/2} \cos x \cos(2n\pi x) \, dx = 2 \left( \frac{\sin((2n\pi - 1) x)}{2n\pi - 1} + \frac{\sin((2n\pi + 1) x)}{2n\pi + 1} \right) \bigg|_{0}^{1/2}
\]

(Our integrand involves the term \( \cos(n\pi x/L) \), since \( L = 1/2 \), we will encounter terms of the form \( \cos(2n\pi x) \)). Note that since the argument of \( \sin \) is \( 2n\pi \) (and not just \( 2n \)) we cannot automatically set \( \sin \) equal to zero at both limits. After some trig identities and algebra we get:

\[
a_n = \frac{-4(-1)^n \sin(1/2)}{4n^2\pi^2 - 1}
\]

Our Fourier Series is then:

\[
f(x) = 2\sin(1/2) + 4\sin(1/2) \left[ \frac{\cos(2\pi x)}{4\pi^2 - 1} - \frac{\cos(4\pi x)}{16\pi^2 - 1} + \frac{\cos(6\pi x)}{36\pi^2 - 1} - \cdots \right]
\]

Verifying through Mathematica:
2. \( f(x) = \begin{cases} -1, & -1 < x < 0 \\ 2, & 0 < x < 2 \end{cases} \)

This function is defined on \([-1, 2]\), thus is \(2L = 3\) periodic, meaning the value of \(L\) we should use is \(3/2\). The Fourier coefficients become:

\[
a_0 = \frac{1}{(3/2)} \left[ \int_{-1}^{0} -1 \, dx + \int_{0}^{2} 2 \, dx \right] = 2
\]

\[
a_n = \frac{1}{(3/2)} \left[ \int_{-1}^{0} \cos \left( 2n \pi \frac{x}{3} \right) \, dx + 2 \int_{0}^{2} \cos \left( 2n \pi \frac{x}{3} \right) \, dx \right] = \frac{2}{3} \left[ \frac{-3}{2n\pi} \sin \left( 2n \pi \frac{x}{3} \right) \bigg|_{-1}^{0} + \frac{3}{n\pi} \sin \left( 2n \pi \frac{x}{3} \right) \bigg|_{0}^{3} \right]^2
\]

\[
= \frac{2}{3} \left[ \frac{-3}{2n\pi} (0 - \sin \left( -2n \pi / 3 \right)) + \frac{2}{n\pi} \sin \left( 4n \pi / 3 \right) \right] = [-\sin \left( 2n \pi / 3 \right) + 2 \sin \left( 4n \pi / 3 \right)] / (n\pi)
\]

Evaluating the first three non-zero coefficients for our Fourier expansion:

\[
a_1 = \frac{1}{\pi} \left( 2 \sin \left( 4 \pi / 3 \right) - \sin \left( 2 \pi / 3 \right) \right) = \frac{1}{\pi} \left( 2 \cdot \left( -\frac{\sqrt{3}}{2} \right) - \frac{\sqrt{3}}{2} \right) = -\frac{3\sqrt{3}}{2\pi}
\]

\[
a_2 = \frac{1}{2\pi} \left( 2 \sin \left( 8 \pi / 3 \right) - \sin \left( 4 \pi / 3 \right) \right) = \frac{1}{2\pi} \left( 2 \cdot \frac{\sqrt{3}}{2} - \left( -\frac{\sqrt{3}}{2} \right) \right) = \frac{3\sqrt{3}}{4\pi}
\]

\[
a_3 = \frac{1}{3\pi} \left( 2 \sin \left( 4 \pi \right) - \sin \left( 2 \pi \right) \right) = 0
\]

\[
a_4 = \frac{1}{4\pi} \left( 2 \sin \left( 16 \pi / 3 \right) - \sin \left( 8 \pi / 3 \right) \right) = \frac{-3\sqrt{3}}{8\pi}
\]
\[ b_n = \frac{1}{(3/2)} \left[ \int_{-1}^{0} -\sin \left( \frac{2n\pi x}{3} \right) \, dx + 2 \int_{0}^{2} \sin \left( \frac{2n\pi x}{3} \right) \, dx \right] = \]

\[
\frac{2}{3} \left[ \frac{3}{2n\pi} \cos \left( \frac{2n\pi x}{3} \right) \bigg|_{-1}^{0} - \frac{3}{n\pi} \cos \left( \frac{2n\pi x}{3} \right) \right]^2 =
\]

\[
\frac{2}{3} \left[ \frac{3}{2n\pi} (1 - \cos \left( \frac{2n\pi}{3} \right)) - \frac{3}{n\pi} (\cos \left( \frac{4n\pi}{3} \right) - 1) \right] =
\]

\[
\frac{1 - \cos \left( \frac{2n\pi}{3} \right)}{n\pi} + \frac{2 (1 - \cos \left( \frac{4n\pi}{3} \right))}{n\pi} = \]

(1)

We will use this expression to evaluate the first three non-zero coefficients:

\[
b_1 = \frac{1}{\pi} (3 - \cos \left( \frac{2\pi}{3} \right) - 2 \cos \left( \frac{4\pi}{3} \right)) = \frac{1}{\pi} (3 - (-1/2) - 2 (-1/2)) = \frac{9}{2\pi}
\]

\[
b_2 = \frac{1}{2\pi} (3 - \cos \left( \frac{4\pi}{3} \right) - 2 \cos \left( \frac{8\pi}{3} \right)) = \frac{1}{2\pi} (3 - (-1/2) - 2 (-1/2)) = \frac{9}{4\pi}
\]

\[
b_3 = \frac{1}{3\pi} (3 - \cos \left( \frac{2\pi}{3} \right) - 2 \cos \left( \frac{12\pi}{3} \right)) = 0
\]

\[
b_4 = \frac{1}{4\pi} (3 - \cos \left( \frac{8\pi}{3} \right) - 2 \cos \left( \frac{16\pi}{3} \right)) = \frac{1}{4\pi} (3 - (-1/2) - 2 (-1/2)) = \frac{9}{8\pi}
\]

Now we can leave the coefficients in the form of eq. (1); this is perfectly acceptable albeit a bit klunky, but perhaps not surprising considering the asymmetric nature of the function. However, if you checked your integral via Mathematica, you likely obtained:

\[
\text{Clear}[f, l]\\
l = 3/2;\\
f = \text{Which}[-1 < x < 0, -1, 0 < x < 2, 2];
\]

(* Notice how we can use the Which function to define piecewise functions *)

(1/1) \(\text{Integrate}[f \sin[n\pi x/1], \{x, -1, 2]\] \)

\[
\frac{2 \left[ \sin[\frac{\pi x}{3}]^2 + 2 \sin[\frac{2\pi x}{3}]^2 \right]}{n\pi}
\]

We can reconcile the two answers using simple trig identities. Combine the double cos formula with the trig version of the Pythagorean theorem and we get:

\[
\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x \Rightarrow 1 - \cos 2x = 2 \sin^2 x
\]

Thus:

\[
\frac{1 - \cos \left( \frac{2n\pi}{3} \right)}{n\pi} + \frac{2 (1 - \cos \left( \frac{4n\pi}{3} \right))}{n\pi} = \frac{2}{n\pi} \left[ \sin \left( \frac{n\pi}{3} \right)^2 + 2 \sin \left( \frac{2n\pi}{3} \right)^2 \right]
\]

Now, with all this at our disposal, the Fourier expansion becomes:
\[ f(x) = 1 - \frac{3\sqrt{3}}{2\pi} \left[ \cos \left( \frac{2\pi x}{3} \right) - \frac{\cos \left( \frac{4\pi x}{3} \right)}{2} + \frac{\cos \left( \frac{8\pi x}{3} \right)}{4} + \ldots \right] + \]
\[ \frac{9}{2\pi} \left[ \sin \left( \frac{2\pi x}{3} \right) + \frac{\sin \left( \frac{4\pi x}{3} \right)}{2} + \frac{\sin \left( \frac{8\pi x}{3} \right)}{4} + \ldots \right] \]

Finally, to verify via Mathematica:

```mathematica
Clear[a0, an, bn]
a0 = 2;
an = (-Sin[2 n \pi / 3] + 2 Sin[4 n \pi / 3]) / (n \pi);
bn = 2 (Sin[n \pi / 3]^2 + 2 Sin[2 n \pi / 3]^2) / (n \pi);
Plot[a0 / 2 + Sum[an Cos[2 n \pi x / 3] + bn Sin[2 n \pi x / 3], {n, 1, 50}], {x, -4, 5}]`
```

And it's a match. And you thought we couldn't expand this function in a Fourier Series. Ye of little faith ...

3. \[ f(x) = \begin{cases} 
-1, & -4 < x < 0 \\
1, & 0 < x < 4 
\end{cases} \]

This is going to be easy. We should be able to determine by inspection that this is an odd function, but since I have Mathematica (literally) at my fingertips, let's:
Clear[f, 1]
1 = 4;
f = Which[-4 < x < 0, -1, 0 < x < 4, 1];
Plot[f, {x, -1, 1}]

Thus we know there will be only sin terms in this expansion, and we can compute:

\[ b_n = \frac{1}{4} \int_{-4}^{4} f(x) \sin(n \pi x / 4) \, dx = \frac{2}{4} \int_{0}^{4} f(x) \sin(n \pi x / 4) \, dx = \]

\[ \frac{1}{2} \int_{0}^{4} \sin(n \pi x / 4) \, dx = \frac{1}{2} \left[ \frac{-4}{n \pi} \cos(n \pi x / 4) \right]_{0}^{4} = \frac{2}{n \pi} (1 - \cos(n \pi)) = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n \pi^2}, & n \text{ odd} \end{cases} \]

The Fourier series is:

\[ f(x) = \frac{4}{\pi} \left[ \sin(\pi x / 4) + \frac{\sin(3 \pi x / 4)}{3} + \frac{\sin(5 \pi x / 4)}{5} + \ldots \right] \]

Verifying:

Plot[(4/\pi) Sum[Sin[n \pi x / 4] / n, {n, 1, 101, 2}], {x, -12, 12}]

4. \[ f(x) = \begin{cases} 1 + 2x, & -2 < x < 0 \\ 1 - 2x, & 0 < x < 2 \end{cases} \]

We should be able to see that this is an even function, but plotting removes all ambiguity:
Thus, there will be no sin terms in this expansion, and we can compute coefficients according to:

\[ a_0 = \frac{2}{2} \int_{0}^{2} (1 - 2x) \, dx = -2 \]
\[ a_n = \frac{2}{2} \int_{0}^{2} (1 - 2x) \cos(n \pi x / 2) \, dx = \frac{2}{n \pi} \sin[n \pi x / 2] \bigg|_0^2 - \frac{1}{n \pi} \cdot 4x \sin[n \pi x / 2] \bigg|_0^2 - \]
\[ \frac{8}{n^2 \pi^2} \cos[n \pi x / 2] \bigg|_0^2 = \frac{8}{n^2 \pi^2} (1 - (-1)^n) = \begin{cases} \frac{16}{n^2 \pi^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \]

Our Fourier series is:

\[ f(x) = -1 + \frac{16}{\pi^2} \left[ \cos(\pi x / 2) + \frac{\cos(3\pi x / 2)}{3^2} + \frac{\cos(5\pi x / 2)}{5^2} + \ldots \right] \]

Verifying:

\[ g_1 = \text{Plot}[-1 + 16/\pi^2 \sum \cos[n \pi x / 2] / n^2, \{n, 1, 101, 2\}, \{x, -6, 6\}] \]
\[ g_2 = \text{Plot}[\text{Which}[-2 < x < 0, 1 + 2x, 0 < x < 2, 1 - 2x], \{x, -2, 2\}, \text{PlotStyle} \rightarrow \text{Red}] \]
\[ \text{Show}[g_1, g_2] \]
5. \( f(x) = e^{-2x}, \ -2 < x < 2 \)

In this problem, \( L = 2 \). The forms of the integrals are well known by now:

\[
a_0 = \frac{1}{2} \int_{-2}^{2} e^{-2x} \, dx = \frac{\sinh(4)}{2}
\]

We can evaluate the \( a \) and \( b \) integrals using the results derived in class:

\[
\left\{ \int e^{ax} \cos bx \, dx, \int e^{ax} \sin bx \, dx \right\} = \frac{e^{ax}}{a^2 + b^2} \{ a \cos(bx) + b \sin(bx), -b \cos(bx) + a \sin(bx) \}
\]

In our integrals, \( a = -2 \) and \( b = n\pi/2 \), therefore:

\[
a_n = \frac{1}{2} \int_{-2}^{2} \cos(n\pi x/2) e^{-2x} \, dx = \frac{e^{-2x}}{16 + n^2 \pi^2} \left( -4 \cos(n\pi x/2) + n\pi \sin(n\pi x/2) \right)_{-2}^{2}
\]

The \( \sin \) terms will go to zero, leaving us with:

\[
a_n = \frac{-4}{16 + n^2 \pi^2} \left( e^{-4 \cos(n\pi)} - e^{-4 \cos(-n\pi)} \right) = \frac{4(-1)^n (e^4 - e^{-4})}{16 + n^2 \pi^2} = \frac{8(-1)^n \sinh(4)}{16 + n^2 \pi^2}
\]

For the \( \sin \) coefficients:

\[
b_n = \frac{1}{2} \int_{-2}^{2} \sin(n\pi x/2) e^{-2x} \, dx = \frac{e^{-2x}}{16 + n^2 \pi^2} \left( -n\pi \cos(n\pi x/2) - 4 \sin(n\pi x/2) \right)_{-2}^{2}
\]

The \( \sin \) terms are zero when evaluated at \( x = 2 \) and \(-2\), leaving us with:

\[
b_n = \frac{-n\pi}{16 + n^2 \pi^2} \left( e^{-4 \cos(n\pi)} - e^{-4 \cos(n\pi)} \right) = \frac{2 \sinh(4) n\pi (-1)^n}{16 + n^2 \pi^2}
\]

We can write the Fourier series as:

\[
f(x) = \frac{\sinh(4)}{4} - 8 \sinh(4) \left[ \frac{\cos(n\pi x/2)}{16 + n^2 \pi^2} - \frac{\cos(2\pi x/2)}{16 + 4 \pi^2} + \frac{\cos(3\pi x/2)}{16 + 9 \pi^2} - \ldots \right]
\]

\[
-2\pi \sinh(4) \left[ \frac{\sin(n\pi x/2)}{16 + n^2 \pi^2} - \frac{2 \sin(2\pi x/2)}{16 + 4 \pi^2} + \frac{3 \sin(3\pi x/2)}{16 + 9 \pi^2} - \ldots \right]
\]

Verifying:
Clear[a0, an, bn]
a0 = Sinh[4] / 2;
an = 8 (-1)^n Sinh[4] / (16 + n^2 π^2);
bn = 2 Sinh[4] n π (-1)^n / (16 + n^2 π^2);
g1 =
    Plot[a0 / 2 + Sum[an Cos[n π x / 2] + bn Sin[n π x / 2], {n, 1, 50}], {x, -6, 6}, PlotRange -> All];
g2 = Plot[Exp[-2 x], {x, -2, 2}, PlotStyle -> Red, PlotRange -> All];
Show[g1, g2, PlotRange -> All]