PHYS 301
HOMEWORK #6-- SOLUTIONS

1. We make successive use of contraction of Kronecker deltas :

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{im} = \delta_{ik}\,\delta_{km}\,\delta_{im} = \delta_{im}\,\delta_{im} = \delta_{im}\,\delta_{mi} = \delta_{ii} = 3$$

The second expression :

$$\epsilon_{ijk}\,\delta_{jk}$$

equals zero. The Kronecker delta term is 0 unless j = k; however, if j = k, the the Levi - Civita permutation tensor is zero. If one term is non - zero, the other term is necessarily zero, so the entire product is always zero.

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2. First, we write the identity in summation notation. Then, we use the product rule to differentiate :

$$\nabla \cdot (f\,\mathbf{g}) \rightarrow \frac{\partial}{\partial x_i}(f\,g_i) = f\,\frac{\partial}{\partial x_i}g_i + g_i\,\frac{\partial f}{\partial x_i}$$

Notice that the next to last term is just f Div g, and the last term is the dot product between g and Grad f. We have then :

$$\nabla \cdot (f\,\mathbf{g}) = f\,\nabla \cdot \mathbf{g} + \mathbf{g} \cdot \nabla f$$

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3. We can apply the results of problem 2 to :

$$\nabla \cdot \left(r^3\,\mathbf{r}\right) = r^3\,\nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla r^3$$

The div of the position vector is simply 3 :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} \Rightarrow \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3$$

Since the scalar magnitude of this vector is :

$$\sqrt{x^2 + y^2 + z^2}$$

We have that :

$$\nabla r^3 = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{3/2}\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{3/2}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{3/2}\,\hat{\mathbf{z}} =$$

$$\frac{3}{2}\left(x^2 + y^2 + z^2\right)^{1/2}(2\,x)\,\hat{\mathbf{x}} + \frac{3}{2}\left(x^2 + y^2 + z^2\right)^{1/2}(2\,y)\,\hat{\mathbf{y}} + \frac{3}{2}\left(x^2 + y^2 + z^2\right)^{1/2}(2\,z)\,\hat{\mathbf{z}} =$$

$$3\left(x^2 + y^2 + z^2\right)^{1/2}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = 3\,r\,\mathbf{r}$$

Combining all these results we get :

$$\nabla \cdot \left(r^3\,\mathbf{r}\right) = 3\,r^3 + \mathbf{r} \cdot (3\,r\,\mathbf{r}) = 3\,r^3 + 3\,r\,\mathbf{r} \cdot \mathbf{r} = 6\,r^3$$

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4. We transform the expression

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{A}) \to A_i\left(\epsilon_{ijk}\,B_j\,A_k\right)$$

Note that the terms in parentheses produce the ith component of the curl of **A×B.** Then

$$A_i\left(\epsilon_{ijk}\,B_j\,A_k\right)_k$$

represents the dot product of **A** and **B × A**. Since all components are scalars, we can rewrite as :

$$A_i\left(\epsilon_{ijk}\,B_j\,A_k\right) = \left(\epsilon_{ijk}\,A_k\,A_i\right)B_j$$

The terms in parentheses now compute the jth component of the cross product between **A** and itself, which we know to be zero.