PHYS 301

HOMEWORK #6-- SOLUTIONS

1. We make successive use of contraction of Kronecker deltas:
\[\delta_{ij} \delta_{jk} \delta_{km} \delta_{im} = \delta_{ik} \delta_{km} \delta_{im} = \delta_{im} \delta_{mi} = \delta_{ii} = 3\]
The second expression:
\[\epsilon_{ijk} \delta_{jk}\]
equals zero. The Kronecker delta term is 0 unless j = k; however, if j = k, the the Levi - Civita permutation tensor is zero. If one term is non-zero, the other term is necessarily zero, so the entire product is always zero.

2. First, we write the identity in summation notation. Then, we use the product rule to differentiate:
\[\nabla \cdot (f \mathbf{g}) \rightarrow \frac{\partial}{\partial x_i} (f g_i) = f \frac{\partial}{\partial x_i} g_i + g_i \frac{\partial f}{\partial x_i}\]
Notice that the next to last term is just f Div g, and the last term is the dot product between g and Grad f. We have then:
\[\nabla \cdot (f \mathbf{g}) = f \nabla \cdot \mathbf{g} + \mathbf{g} \cdot \nabla f\]

3. We can apply the results of problem 2 to:
\[\nabla \cdot (r^3 \mathbf{r}) = r^3 \nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla r^3\]
The div of the position vector is simply 3:
\[\mathbf{r} = x \hat{x} + y \hat{y} + z \hat{z} \Rightarrow \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3\]
Since the scalar magnitude of this vector is:
\[\sqrt{x^2 + y^2 + z^2}\]
We have that:
\[\nabla r^3 = \frac{\partial}{\partial x} \left( x^2 + y^2 + z^2 \right)^{3/2} \hat{x} + \frac{\partial}{\partial y} \left( x^2 + y^2 + z^2 \right)^{3/2} \hat{y} + \frac{\partial}{\partial z} \left( x^2 + y^2 + z^2 \right)^{3/2} \hat{z} = \]
\[\frac{3}{2} \left( x^2 + y^2 + z^2 \right)^{1/2} (2x) \hat{x} + \frac{3}{2} \left( x^2 + y^2 + z^2 \right)^{1/2} (2y) \hat{y} + \frac{3}{2} \left( x^2 + y^2 + z^2 \right)^{1/2} (2z) \hat{z} = \]
\[3 \left( x^2 + y^2 + z^2 \right)^{1/2} (x \hat{x} + y \hat{y} + z \hat{z}) = 3 \mathbf{r} \cdot \mathbf{r}\]
Combining all these results we get:
\[\nabla \cdot (r^3 \mathbf{r}) = 3 r^3 + \mathbf{r} \cdot (3 r \mathbf{r}) = 3 r^3 + 3 \mathbf{r} \cdot \mathbf{r} = 6 r^3\]
4. We transform the expression
\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) \rightarrow A_i \left( \epsilon_{ijk} B_j A_k \right) \]

Note that the terms in parentheses produce the ith component of the curl of \( \mathbf{A} \times \mathbf{B} \). Then
\[ A_i \left( \epsilon_{ijk} B_j A_k \right)_k \]

represents the dot product of \( \mathbf{A} \) and \( \mathbf{B} \times \mathbf{A} \). Since all components are scalars, we can rewrite as:
\[ A_i \left( \epsilon_{ijk} B_j A_k \right) = \left( \epsilon_{ijk} A_k A_i \right) B_j \]

The terms in parentheses now compute the jth component of the cross product between \( \mathbf{A} \) and itself, which we know to be zero.