PHYS 301
HOMEWORK #8

Due : 19 March 2014

1. Consider the vector force function :

$$\mathbf{F} = y^2 z^3 \hat{\mathbf{x}} + 2 x y z^3 \hat{\mathbf{y}} + 3 x y^2 z^2 \hat{\mathbf{z}}$$

a) Determine whether the force is conservative. (5)

Solution : We determine whether a vector field is conservative by computing its curl :

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 z^3 & 2 x y z^3 & 3 x y^2 z^2 \end{vmatrix} =$$

$$\left(6 x y z^2 - 6 x y z^2\right) \hat{\mathbf{x}} - \left(3 y^2 z^2 - 3 y^2 z^2\right) \hat{\mathbf{y}} + \left(2 y z^3 - 2 y z^3\right) \hat{\mathbf{z}} = \{0, 0, 0\}$$

b) If the force is conservative, determine the scalar potential which generates it. (10)

Solution : We know that the force can be generated from a scalar potential according to :

$$\mathbf{F} = -\nabla \phi = -\left(\frac{\partial \phi}{\partial x} \hat{\mathbf{x}} + \frac{\partial \phi}{\partial y} \hat{\mathbf{y}} + \frac{\partial \phi}{\partial z} \hat{\mathbf{z}}\right)$$

Thus, we can equate each component of **F** to each component of $\nabla \phi$. (For the sake of sanity, we will ignore the minus sign until the very end).Starting with the x component we get :

$$\frac{\partial \phi}{\partial x} = y^2 z^3 \Rightarrow \phi = x y^2 z^3 + g(y, z)$$

Taking the partial derivative of this expression for ϕ with respect to y yields an expression that must equal the y component of **F** :

$$\frac{\partial \phi}{\partial y} = 2 x y z^3 + \frac{\partial g(y, z)}{\partial y} = 2 x y z^3 \Rightarrow \frac{\partial g(y, z)}{\partial y} = 0 \Rightarrow g = g(z)$$

Taking the derivative of ϕ with respect to z and equating to the z component of the force gives :

$$\frac{\partial \phi}{\partial z} = 3 x y^2 z^2 + \frac{dg}{dz} = 3 x y^2 z^2 \Rightarrow g = \text{constant}$$

Thus, the scalar potential that generates this function is :

$$\phi = -3 x y^2 z^2 + \text{constant}$$

c) Compute the work done by this force if it acts between (0, 0, 0) and (1, 2, 3). (10)

Solution : For a conservative force, the work done is equal to the difference in the potential at the endpoints, or :

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_a^b -\nabla\phi \cdot d\mathbf{l} = \int_a^b -d\phi = -(\phi(b) - \phi(a)) = 3(1)(2)^2(3)^2 - 0$$

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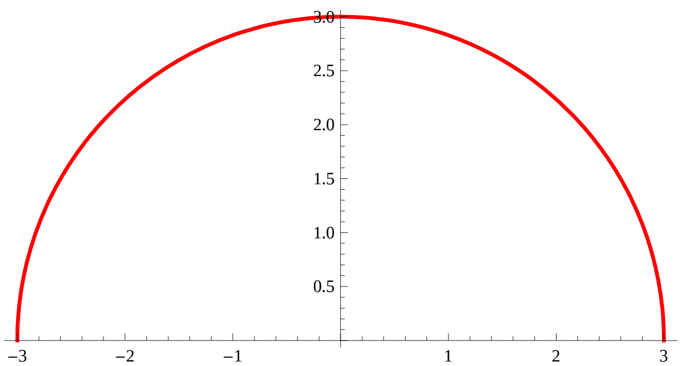
2. Consider the force : (10 pts for each part)

$$\mathbf{F} = e^x \cos y\, \hat{\mathbf{x}} - e^x \sin y\, \hat{\mathbf{y}}$$

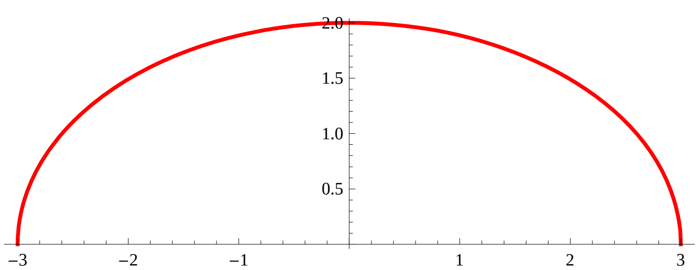
Compute the work done by this force in acting along the path:

a) the upper semicircle of the circle of radius 3 centered on the origin (going from - 3 to 3) :



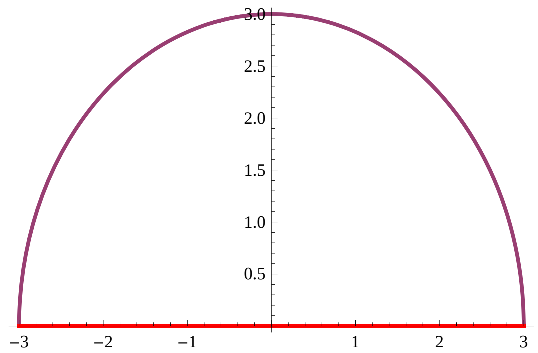
b) The upper half of the ellipse (also going from - 3 to 3) :

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



(Hint : ***Think*** before you feel the need to solve very elaborate integrals)

Solutions : If you tried to parameterize the force and the path, you probably found the resulting integrals very challenging. The thinking way to solve both of these integrals would be to notice that F is a conservative function, so that the line integral of F chosen over any closed path would be zero. Thus, let' s consider the path :



where we create a closed path by including the straight line connecting (3, 0) and (-3, 0). Thus, we have that :

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int_{\text{curve}} \mathbf{F} \cdot d\mathbf{l} + \int_{3}^{-3} F_x \, dx = 0$$

All we need to do in both cases is find the value of the line integral from - 3 to 3 along the x axis, and we know that is the negative of the line integral along the curve. Since y = 0 = dy along the x axis (so cos y = 1), the line integral is quite simple :

$$\int_{3}^{-3} F_x \, dx = \int_{3}^{-3} e^x \, dx = e^{-3} - e^3 = -2 \sinh 3$$

Thus, the value of the line integrals in both parts is 2 sinh 3.

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3. If **r** is the position vector, compute

$$\int_S \mathbf{r} \cdot \mathbf{n} \, da$$

where the surface is : a) the surface of a sphere of radius 3 centered on the origin, and b) a cube of length L. (10 pts each part)

Solution : We use the divergence theorem :

$$\int_S \mathbf{r} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{r} \, d\tau$$

Since **r** is the position vector, we know that $\nabla \cdot r = 3$, so that the value of the surface integral is simply three times the volume of each figure. For part a) the integral becomes $3 * \left(\frac{4}{3} \pi 3^3\right) = 108 \pi$. For part b, the answer is simply $3 L^3$,

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4. Prove that :

$$\int_V (\nabla \times \mathbf{v})\, d\tau = -\int_S \mathbf{v} \times \mathbf{da}$$

Hint : Replace **v** by **(v × c)** in the divergence theorem, where c is a constant vector.

Solution : We begin with the divergence theorem :

$$\int_S \mathbf{v} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{v}\, d\tau$$

Now, let **v** → **v × c** and we get :

$$\int_S (\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a} = \int_V \nabla \cdot (\mathbf{v} \times \mathbf{c})\, d\tau$$

We employ Einstein summation notation for both sides of this equation. On the left, we can write :

$$(\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a} = (\epsilon_{ijk}\, v_j\, c_k)\, da_i = -(\epsilon_{ijk}\, v_j\, da_i)\, c_k = -(\mathbf{v} \times d\mathbf{a}) \cdot \mathbf{c}$$

The minus sign arises since our permutation is with respect to the triad ' i j k', and **j × i = -k**.

On the right, we can write :

$$\nabla \cdot (\mathbf{v} \times \mathbf{c}) = \frac{\partial}{\partial x_i} (\epsilon_{ijk}\, v_j\, c_k) = \left(\epsilon_{ijk} \frac{\partial}{\partial x_i} v_j\right) c_k = (\nabla \times \mathbf{v}) \cdot \mathbf{c}$$

Using these results, the initial identity becomes :

$$\int_S -(\mathbf{v} \times d\mathbf{a}) \cdot \mathbf{c} = \int_V (\nabla \times \mathbf{v}) \cdot \mathbf{c}\, d\tau$$

Now, take the dot product of both sides with **c** and divide by c^2, and you have shown the requested result.