1. The function:

\[ f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
x, & 0 < x < \pi 
\end{cases} \]

has the Fourier series:

\[ f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[ \cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \ldots \right] + \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots \right] \]

We are asked to use Dirichlet's Theorem to derive expressions for \( \pi \) using the values of \( f \) at \( x = 0 \), \( x = \pi/2 \) and \( x = \pi \). Remember that Dirichlet's theorem states that if a function can be expressed as a Fourier series, then the Fourier series converges to the value of \( f \) where \( f \) is continuous, and to the midpoint of a discontinuity. For this \( 2\pi \) periodic function (I have plotted three cycles of this function):

we see that \( f \) is continuous at \( x = 0 \) and \( x = \pi/2 \), but is discontinuous at \( x = \pi \). Since \( f(0) = 0 \), we have:

\[ f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi} \left[ \cos 0 + \frac{\cos 0}{9} + \frac{\cos 0}{25} + \ldots \right] + \left[ \sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} - \ldots \right] \]

Since \( \cos 0 = 1 \) and \( \sin 0 = 0 \), we obtain:

\[ 0 = \frac{\pi}{4} - \frac{2}{\pi} \left[ 1 + \frac{1}{9} + \frac{1}{25} + \ldots \right] \Rightarrow \frac{\pi^2}{8} = \sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} \]

Let's check this with Mathematica:
Sum\[1/n^2,\{n,1,\infty,2\}\]

\[\frac{\pi^2}{8}\]

Now, at \(x = \frac{\pi}{2}\), \(f(x) = \frac{\pi}{2}\) and the series converges to this value, so :

\[
f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = \frac{\pi}{4}
\]

\[
\frac{2}{\pi} \left[ \cos \frac{\pi}{2} + \cos \left(\frac{3\pi}{2}\right) \frac{9}{25} + \cos \left(\frac{5\pi}{2}\right) + \cdots \right] + \left[ \sin \frac{\pi}{2} - \sin \left(\frac{\pi}{2}\right) + \sin \left(\frac{3\pi}{2}\right) \frac{9}{3} + \cdots \right]
\]

In this case, all the cos terms are zero since \(\cos \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}\), etc is zero. The sin terms vanish for even values of \(n\), and the sign of the odd terms alternate, so we have :

\[
\frac{\pi}{2} = 4 + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

Verifying through Mathematica :

\[
\text{Sum}[(-1)^n \sin[n \frac{\pi}{2}]/n,\{n,1,\infty,2\}]
\]

\[\frac{\pi}{4}\]

Finally, at \(x = \pi\) we have to realize that there is a discontinuity whose midpoint is \(\pi/2\), therefore, we have :

\[
f(\pi) = \frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi} \left[ \cos \pi + \cos \left(\frac{3\pi}{2}\right) \frac{9}{25} + \cos \left(\frac{5\pi}{2}\right) + \cdots \right] + \text{[a series of sin (n \pi) terms]}
\]

we know the sin \((n \pi)\) terms are all zero, and that \(\cos (n \pi) = -1\) for all odd values of \(n\), so Dirichlet's theorem gives us :

\[
\frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi} \left[ -1 + \frac{1}{3} - \frac{1}{5} + \cdots \right] \Rightarrow \frac{\pi^2}{8} = \sum_{\text{odd } n} \frac{1}{n^2}
\]

as we found in part a).

2. We are asked to find two Fourier series. In both cases, the function is \(2L\) periodic on the given interval. Since the function is \(2L\) periodic, we do not need to extend the function (as we will in problems from section 9). We are asked to find the Fourier series for \(f(x) = 1 + x\) on \((0, 4)\) and then on \((-2, 2)\). In both cases, the interval is 4, so \(2L = 4\) and \(L = 2\). To find the Fourier coefficients (and then the series) on \((0, 4)\), we write :

\[
a_0 = \frac{1}{2} \int_0^4 (1 + x) \cos (nx/2) \, dx = 0
\]

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Since the interval is \((0, 4)\), we cannot make use of symmetry arguments. For the other coefficients we have :

\[
a_n = \frac{1}{2} \int_0^4 (1 + x) \cos (nx/2) \, dx = 0
\]
\[ b_n = \frac{1}{2} \int_{0}^{4} (1 + x) \sin \left( \frac{n \pi x}{2} \right) \, dx = \frac{-4}{n \pi} \]

so that the Fourier expansion becomes:

\[ f(x) = 3 - \frac{4}{\pi} \left[ \sin \left( \frac{\pi x}{2} \right) + \frac{\sin(\pi x)}{2} + \frac{\sin(3\pi x/2)}{3} + \ldots \right] \]

And plotting this series over three cycles:

\[ \text{Plot}[3 - (4/\pi) \sum[\sin(n \pi x/2) / n, \{n, 1, 31\}, \{x, -4, 8\}] \]

For part b), our value of L = 2 still, but our limits of integration are between (-2, 2), so we obtain:

\[ a_0 = \frac{1}{2} \int_{-2}^{2} (1 + x) \, dx = 2 \]

\[ a_n = \frac{1}{2} \int_{-2}^{2} (1 + x) \cos \left( \frac{n \pi x}{2} \right) \, dx \]

Even though \( f(x) \) is neither even nor odd, we can break the function into an even piece (here, simply 1) and an odd piece (x). We know the integral of \( x \cos \left( \frac{n \pi x}{2} \right) \) will be zero on (-2, 2) since the integrand is odd. Therefore, the only non-zero part of this integral is:

\[ a_n = \frac{1}{2} \int_{-2}^{2} \cos \left( \frac{n \pi x}{2} \right) \, dx = \frac{2}{2} \int_{0}^{2} \cos \left( \frac{n \pi x}{2} \right) \, dx = \frac{2}{n \pi} \sin \left( \frac{n \pi x}{2} \right) \bigg|_{0}^{2} = 0 \]

Computing the b coefficients (and employing symmetry to simplify the integral):

\[ b_n = \frac{1}{2} \int_{-2}^{2} (1 + x) \sin \left( \frac{n \pi x}{2} \right) \, dx = \frac{2}{2} \int_{0}^{2} x \sin \left( \frac{n \pi x}{2} \right) \, dx = \frac{-4 (-1)^n}{n \pi} \]

Our Fourier series is:

\[ f(x) = 1 + \frac{4}{\pi} \left[ \sin \left( \frac{\pi x}{2} \right) - \frac{\sin(\pi x)}{2} + \frac{\sin(3\pi x/2)}{3} - \ldots \right] \]

Plotting three cycles:
3. Find the Fourier series for \( f(x) = x \)
   a) on \((-\pi, \pi)\):
   We can make use of symmetry here since the function is odd. Therefore,
   
   \[
   a_0 = a_n = 0, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin (n \pi x) \, dx = -2 \frac{(-1)^n}{n}
   \]
   And our series is:
   
   \[
   f(x) = 2 \left[ \sin x - \frac{\sin (2x)}{2} + \frac{\sin (3x)}{3} + \ldots \right]
   \]
   Plotting three cycles:
   
   ![Plot showing three cycles of the function](image)

   b) on \((-3, 3)\) We can still use symmetry, but must remember to use the proper form of the Fourier coefficients and series:
   
   Because of the odd symmetry of the function, all the a coefficients are zero, and:
   
   \[
   b_n = \frac{2}{3} \int_{0}^{3} x \sin (n \pi x / 3) \, dx = -\frac{6}{n \pi} (-1)^n
   \]
The Fourier series is:

\[ f(x) = \frac{6}{\pi} \left[ \sin(\pi x/3) - \frac{\sin(2 \pi x/3)}{2} + \frac{\sin(3 \pi x/3)}{3} + \ldots \right] \]

Plotting three cycles:

Plot((-6/\pi) \sum[(-1)^n \sin[n \pi x/3] / n, \{n, 1, 41\}], \{x, -6, 6\})

4. \( f(x) = \left\{ \begin{array}{ll}
-1, & -1 < x < 0 \\
1, & 0 < x < 3 
\end{array} \right. \)

This function is 2L periodic on (-1, 3). This means that L = 2. To find our Fourier coefficients, we compute:

\[ a_0 = \frac{1}{2} \int_{-1}^{3} f(x) \, dx = \frac{1}{2} \left[ \int_{0}^{3} dx + \int_{-1}^{0} (-1) \, dx \right] = 1 \]

\[ a_n = \frac{1}{2} \int_{-1}^{3} f(x) \cos(n \pi x/2) \, dx = \frac{1}{2} \left[ \int_{0}^{3} \cos(n \pi x/2) \, dx + \int_{-1}^{0} (-1) \cos(n \pi x/2) \, dx \right] \]

\[ = \frac{1}{n \pi} (\sin(3n \pi/2) - \sin(n \pi/2)) = \left\{ \begin{array}{ll}
0, & n \text{ even} \\
\frac{-2}{n \pi}, & n = 1, 5, 9, \ldots \\
\frac{2}{n \pi}, & n = 3, 7, 11, \ldots 
\end{array} \right. \]

\[ b_n = \frac{1}{2} \left[ \int_{0}^{3} \sin(n \pi x/2) \, dx + \int_{-1}^{0} (-1) \sin(n \pi x/2) \, dx \right] \]

\[ = \frac{1}{2} \left[ \left. -\frac{2}{n \pi} \cos(n \pi x/2) \right|_{0}^{3} + \left. \frac{2}{n \pi} \cos(n \pi x/2) \right|_{-1}^{0} \right] \]

\[ = \frac{1}{n \pi} \left[ -(\cos(3n \pi/2) - 1) + (1 - \cos(n \pi/2)) \right] \]

\[ = \frac{1}{n \pi} [2 - \cos(3n \pi/2) - \cos(n \pi/2)] = \left\{ \begin{array}{ll}
\frac{2}{n \pi}, & n \text{ odd} \\
\frac{4}{n \pi}, & n = 2, 6, 10, \ldots \\
0, & n = 4, 8, 12, \ldots 
\end{array} \right. \]
The Fourier series is:

\[
f(x) = \frac{1}{2} - \frac{2}{\pi} \left[ \cos \left( \frac{\pi x}{2} \right) - \frac{\cos \left( \frac{3\pi x}{2} \right)}{3} + \frac{\cos \left( \frac{5\pi x}{2} \right)}{5} - \ldots \right] + \frac{2}{\pi} \left[ \sin \left( \frac{\pi x}{2} \right) + \frac{\sin \left( \frac{2\pi x}{2} \right)}{2} + \frac{\sin \left( \frac{3\pi x}{2} \right)}{3} + \frac{\sin \left( \frac{5\pi x}{2} \right)}{5} + \frac{2 \sin \left( \frac{6\pi x}{2} \right)}{6} + \ldots \right]
\]

5. Plot the Fourier series from problem 4:

\[
\text{In[42]}:= \text{Plot}\left[ \frac{1}{2} + \frac{1}{\pi} \sum\left( (\sin[3n\pi/2] - \sin[n\pi/2]) \cos[n\pi x/2] / n + (2 - \cos[3n\pi/2] - \cos[n\pi/2]) \sin[n\pi x/2] / n, \{n, 1, 31\}, \{x, -5, 9\} \right) \right]
\]

\[
\text{Out[42]}= \text{Plot}\left[ \frac{1}{2} + \frac{1}{\pi} \sum\left( (\sin[3n\pi/2] - \sin[n\pi/2]) \cos[n\pi x/2] / n + (2 - \cos[3n\pi/2] - \cos[n\pi/2]) \sin[n\pi x/2] / n, \{n, 1, 31\}, \{x, -5, 9\} \right) \right]
\]

6. Every three months, the existing amount of money receives 1% interest; after the interest is applied, you deposit $100.

(* Do Loop *)

Clear[money, interest, deposit]
money=1000.0; deposit=100;
interest=0.01;
Do[money=(1+interest) money+deposit,\{n,40\}]
Print["At the end of ten years you will have ",money," in the bank"]

At the end of ten year you will have $6377.5 in the bank

Each value of n represents a three month period; in 10 years there are 120 months and so 40 accumulation periods.

(* For statement *)

Clear[money, interest, deposit]
deposit=100; interest=0.01;
For[money=1000;n=1,n<41,n++,money=(1+interest) money+deposit]
Print["At the end of ten years you will have ",money," in the bank"]

At the end of ten years you will have $6377.5 in the bank
Clear[money, interest, deposit]
money = 1000; deposit = 100; interest = 0.01; n = 1;
While[n < 41, money = (1 + interest) money + deposit; n++]
Print["At the end of ten years you will have ", money, " in the bank"]

At the end of ten years you will have $6377.5 in the bank