PHYS 301

HOMEWORK #5-- Solutions

1. For the first part of the question, we repeatedly contract Kronecker deltas:
\[ \delta_{ij} \delta_{jk} \delta_{km} \delta_{im} = \delta_{ik} \delta_{km} \delta_{im} = \delta_{im} \delta_{mi} = \delta_{mm} = 3 \]

For the second, we recognize that we have three repeated indices, i, j, and k, so we are summing over all three indices. Further, we recall that the Levi-Civita permutation tensor is zero unless all three indices are different. Thus, we can write our expression as:
\[ \epsilon_{ijk} \epsilon_{ijk} = \epsilon_{111} \epsilon_{111} + \epsilon_{112} \epsilon_{112} + \epsilon_{113} \epsilon_{113} + \ldots \]
and if we were to write out every term explicitly, we would have 27 terms on the right. However, we know that only six terms will be non-zero:
\[ \epsilon_{ijk} \epsilon_{ijk} = \epsilon_{123} \epsilon_{123} + \epsilon_{132} \epsilon_{132} + \epsilon_{213} \epsilon_{213} + \epsilon_{231} \epsilon_{231} + \epsilon_{312} \epsilon_{312} + \epsilon_{321} \epsilon_{321} \]
Each product on the right is either (1) (1) or (-1) (-1), so that the sum of all the terms is 6.

We can also make use of the \( \epsilon - \delta \) relationship. Expanding with respect to the subscript i:
\[ \epsilon_{ijk} \epsilon_{ijk} = \delta_{ji} \delta_{kk} - \delta_{jk} \delta_{kj} = 3 \cdot 3 - \delta_{jj} = 3 \cdot 3 - 3 = 6 \]

2. We translate our identity into Einstein summation notation:
\[ \nabla \cdot (f \mathbf{g}) = \frac{\partial}{\partial x_i} (f g_i) \]
Remember, \( f \) is a scalar and has no components (so will not have any subscripts). Applying the product rule to this expression:
\[ \frac{\partial}{\partial x_i} (f g_i) = f \frac{\partial}{\partial x_i} g_i + g_i \frac{\partial}{\partial x_i} f \]
The first term on the right is the scalar \( f \) multiplied by the dot product between the del operator and \( g \), or in other words: \( f \nabla \cdot \mathbf{g} \)
The second term is the dot product of \( g \) with \( \nabla f \), so our identity is:
\[ \nabla \cdot (f \mathbf{g}) = \frac{\partial}{\partial x_i} (f g_i) = f \frac{\partial}{\partial x_i} g_i + g_i \frac{\partial}{\partial x_i} f = f \nabla \cdot \mathbf{g} + \mathbf{g} \cdot \nabla f \]

3. Using the identity from problem 2, we have:
\[ \nabla \cdot (r^3 \mathbf{r}) = r^3 \nabla \cdot \mathbf{r} + \mathbf{r} \cdot (\nabla r^3) \]

The divergence of \( \mathbf{r} \) is simply:
\[ \nabla \cdot \mathbf{r} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3 \]
To find the gradient of the scalar $r^3$, we first write:

$$r = |r| = \sqrt{x^2 + y^2 + z^2}$$

so that

$$r^3 = (x^2 + y^2 + z^2)^{3/2}$$

and:

$$\nabla r^3 = \frac{\partial}{\partial x} \left( x^2 + y^2 + z^2 \right)^{3/2} \hat{x} + \frac{\partial}{\partial y} \left( x^2 + y^2 + z^2 \right)^{3/2} \hat{y} + \frac{\partial}{\partial z} \left( x^2 + y^2 + z^2 \right)^{3/2} \hat{z}$$

$$= \frac{3}{2} (2x) \sqrt{x^2 + y^2 + z^2} \hat{x} + \frac{3}{2} (2y) \sqrt{x^2 + y^2 + z^2} \hat{y} + \frac{3}{2} (2z) \sqrt{x^2 + y^2 + z^2} \hat{z}$$

$$= 3 \left( \sqrt{x^2 + y^2 + z^2} \right) (x \hat{x} + y \hat{y} + z \hat{z}) = 3rr$$

Substituting these results into the original equation (1):

$$\nabla \cdot (r^3 r) = 3r^3 + r \cdot (3rr) = 3r^3 + 3rr^2 = 6r^3$$

4. Consider:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A})$$

We know that $\mathbf{B} \times \mathbf{A}$ will produce a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Therefore, we expect a vanishing dot product for a vector $\mathbf{A}$ and a vector perpendicular to $\mathbf{A}$. Let's see if we can reproduce this result using summation notation. First, transform the expression to summation notation:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) \rightarrow A_i (\epsilon_{ijk} B_j A_k) = B_j \epsilon_{ijk} A_k A_i \rightarrow \mathbf{B} \cdot (\mathbf{A} \times \mathbf{A})$$

At this point, you can successfully argue that any vector crossed with itself is zero since the angle between them is zero.