PHYS 301
HOMEWORK #1
Solutions
Due: 22 Jan 2016

Homework is due in class on the due date noted. Review carefully the syllabus for proper format (complete, legible solutions; write on only one side of the paper, staple multiple sheets).

1. The transformation equations between the Cartesian and spherical polar coordinate systems are:

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

where \( r \) is the distance from the origin, \( \theta \) is the polar angle and \( \phi \) is the azimuthal angle. Use these equations to determine expressions for the total differentials \( dx, dy \) and \( dz \) in terms of \( r, \theta, \phi, dr, d\theta \) and \( d\phi \).

**Solution**: Recall from multivariable calculus that the total differential of a function of several variables is:

\[ df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \]

Applying this definition to our transformation equations yields:

\[ dx = \sin \theta \cos \phi \, dr + r \cos \theta \cos \phi \, d\theta - r \sin \theta \sin \phi \, d\phi \]
\[ dy = \sin \theta \sin \phi \, dr + r \cos \theta \sin \phi \, d\theta + r \sin \theta \cos \phi \, d\phi \]
\[ dz = \cos \theta \, dr - r \sin \theta \, d\theta \]

2. The Pythagorean theorem can be written in terms of the incremental distance between any two points in space as:

\[ ds^2 = dx^2 + dy^2 + dz^2 \]

where \( ds \) is the differential distance between any two points. Use the expressions for \( dx, dy \) and \( dz \) above to write \( ds^2 \) in terms of spherical polar coordinates. What are the coefficients of the mixed terms \( (dr \, d\theta, dr \, d\phi \) and \( d\theta \, d\phi) \)?

**Solution**: You can always plug and chug and combine terms. We can try a slightly more elegant approach by realizing the square of a trinomial is:

\[ (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) \]

This tells us that three of our terms will involve the squares of the differentials, i.e., terms in \((dr)^2\), \((d\theta)^2\) and \((d\phi)^2\), and these terms will be positive (since the square of a real number is always pos-
tive). You should find that the mixed terms all equal zero. As an example, let’s find the coefficient for the $dr \, d\theta$; multiplying appropriate factors we get:

$$2 \left( \sin \theta \cos \phi \cdot r \cos \theta \cos \phi + r \sin \theta \cos \theta \sin^2 \phi - r \sin \theta \cos \phi \right) dr \, d\theta$$

The expression in parentheses becomes:

$$r \sin \theta \cos \theta \cos^2 \phi + r \sin \theta \cos \theta \sin^2 \phi - r \sin \theta \cos \phi =$$

$$r \sin \theta \cos \theta - r \sin \theta \cos \phi = 0.$$ 

Expanding the other mixed terms and using trig identities will similarly produce mixed terms of zero. Even though we have not formally started our study of Mathematica, let me write a short Mathematica program that will demonstrate the utility and power of the language:

```mathematica
Clear[x,y,z,r,\[Theta],\[Phi],dx,dy,dz]
x = r Sin[\[Theta]]Cos[\[Phi]]; 
y = r Sin[\[Theta]] Sin[\[Phi]]; 
z = r Cos[\[Theta]]; 
dx = D[x,r]dr+D[x,\[Theta]]d\[Theta]+D[x,\[Phi]]d\[Phi]; 
dy=D[y,r]dr+D[y,\[Theta]]d\[Theta]+D[y,\[Phi]]d\[Phi]; 
dz = D[z,r]dr+D[z,\[Theta]]d\[Theta]; 
Collect[dx^2+dy^2+dz^2,(dr,d\[Theta],d\[Phi]),Simplify]
```

$$dr^2 + d\theta^2 \, r^2 + dr^2 \, r^2 \sin^2 \theta$$

The output line shows that the only surviving terms are the squares of the differentials, and that the coefficients of these terms are 1, $r^2$ and $r^2 \sin^2 \theta$. This is actually a klunky way to write this code; as we learn more about Mathematica I can show you more elegant approaches.

3. Consider the function

$$f = x^2 + y^2 + z^2$$

Find:

a) \( \nabla f \) (grad \( f \))

b) \( \nabla \cdot (\nabla f) \) (div grad \( f \))

c) \( \nabla \times (\nabla f) \) (curl grad \( f \))

**Solution** : We make use of the definitions of grad, div and curl in Cartesian coordinates:

a) \( \nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} = 2x \hat{x} + 2y \hat{y} + 2z \hat{z} = 2 \mathbf{r} \)

where \( \mathbf{r} \) is the position vector.

b) If \( \mathbf{v} \) is a vector with components in the \( x \), \( y \) and \( z \) directions, then div \( \mathbf{v} \) is:

$$b) \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$
In our case, \( \text{div} \ (\text{grad} \ f) \) is:

\[
\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (2y) + \frac{\partial}{\partial z} (2z) = 6
\]

c) The curl of a vector can be computed from the determinant:

\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
v_x & v_y & v_z
\end{vmatrix}
\]

In the present case, the curl becomes:

\[
\nabla \times (\nabla f) = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x & 2y & 2z
\end{vmatrix} = 0
\]

We will learn that it is a general result that the curl of any gradient is zero; this result is important in the study of vector fields.

4. Compute the line integral for the function

\[
f = -y \hat{x} + x \hat{y}
\]

along the circle of radius 3 centered on the origin. The symbols \( \hat{x} \) and \( \hat{y} \) indicate unit vectors in the \( x \) and \( y \) directions respectively.

**Solution**: The line integral of a function around a contour is given by:

\[
\int_C \mathbf{f} \cdot d\mathbf{l}
\]

where we integrate the dot product of \( \mathbf{f} \) and \( d\mathbf{l} \), and \( d\mathbf{l} \), the element of arc length, is written in Cartesian coordinates as:

\[
d\mathbf{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}
\]

Since this integral lies in the \( x \)-\( y \) plane, \( z = dz = 0 \), so we have:

\[
\int_C (f_x \, dx + f_y \, dy)
\]

You should recall from multivariable calculus that when your contour is a circle, the best approach is to use the parameterization:

\[
x = r \cos \theta \quad y = r \sin \theta
\]

Then, \( dx \) and \( dy \) become:

\[
dx = -r \sin \theta \, d \theta \quad dy = r \cos \theta \, d \theta
\]

(Remember that since our contour is a circle the radius is constant so that \( dr = 0 \).) With parameterization, the line integral becomes:

\[
\int_C \mathbf{f} \cdot d\mathbf{l} = \int_0^{2\pi} [-r \sin \theta (-r \sin \theta \, d\theta) + r \cos \theta (r \cos \theta \, d\theta)] =
\]
\[ \int_0^{2\pi} \left( r^2 \sin^2 \theta + r^2 \cos^2 \theta \right) d\theta = \int_0^{2\pi} r^2 \, d\theta \]

Since \( r \) is constant along the circle, this simply yields the value \( 2 \pi r^2 \). Since \( r = 3 \) in this case, the line integral is \( 18\pi \).