

PHYS 301

HOMEWORK #2--Solutions

Question #1:

The transformation equations from Cartesian to spherical polar coordinates are :

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

a) Scale Factors

We begin by constructing an expression for $(ds)^2$ by computing the values of dx , dy and dz :

$$dx = \sin \theta \cos \phi \, dr + r \cos \theta \cos \phi \, d\theta - r \sin \theta \sin \phi \, d\phi$$

$$dy = \sin \theta \sin \phi \, dr + r \cos \theta \sin \phi \, d\theta + r \sin \theta \cos \phi \, d\phi$$

$$dz = \cos \theta \, dr - r \sin \theta \, d\theta$$

We square each of these terms; I will write these expressions with the perfect squares first followed by the cross - terms :

$$\begin{aligned} (dx)^2 &= \sin^2 \theta \cos^2 \phi (dr)^2 + r^2 \cos^2 \theta \cos^2 \phi (d\theta)^2 + r^2 \sin^2 \theta \sin^2 \phi (d\phi)^2 \\ &\quad + 2 r \sin \theta \cos \theta \cos^2 \phi \, dr \, d\theta - 2 r \sin^2 \theta \cos \phi \sin \phi \, dr \, d\phi - 2 r^2 \cos \theta \sin \theta \cos \phi \sin \phi \, d\theta \, d\phi \end{aligned}$$

$$\begin{aligned} (dy)^2 &= \sin^2 \theta \sin^2 \phi (dr)^2 + r^2 \cos^2 \theta \sin^2 \phi (d\theta)^2 + r^2 \sin^2 \theta \cos^2 \phi (d\phi)^2 + \\ &\quad 2 r \sin \theta \cos \theta \sin^2 \phi \, dr \, d\theta + 2 r \sin^2 \theta \sin \phi \cos \phi \, dr \, d\phi + 2 r^2 \cos \theta \sin \theta \sin \phi \cos \phi \, d\theta \, d\phi \end{aligned}$$

$$(dz)^2 = \cos^2 \theta (dr)^2 + r^2 \sin^2 \theta (d\theta)^2 - 2 r \cos \theta \sin \theta \, dr \, d\theta.$$

Now, we add each of these expressions, and group like terms to obtain :

$$(dx)^2 + (dy)^2 + (dz)^2 =$$

$$\begin{aligned}
& (dr)^2 [\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta] + \\
& r^2 (d\theta)^2 [\cos^2 \theta \cos^2 \phi + \sin^2 \theta \cos^2 \phi + \sin^2 \theta] + r^2 (d\phi)^2 [\sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi] + \\
& dr d\theta [2 r \sin \theta \cos \theta \cos^2 \phi + 2 r \sin \theta \cos \theta \sin^2 \phi - 2 r \cos \theta \sin \theta] + \\
& dr d\phi [-2 r \sin^2 \theta \cos \phi \sin \phi dr d\phi + 2 r \sin^2 \theta \sin \phi \cos \phi dr d\phi] + \\
& d\theta d\phi [-2 r^2 \cos \theta \sin \theta \cos \phi \sin \phi d\theta d\phi + 2 r^2 \cos \theta \sin \theta \sin \phi \cos \phi d\theta d\phi] = \\
& (dr)^2 [\sin^2 \theta + \cos^2 \theta] + r^2 (d\theta)^2 [\cos^2 \theta + \sin^2 \theta] + r^2 (d\phi)^2 [\sin^2 \theta (\sin^2 \phi + \cos^2 \phi)] - \\
& 2 r dr d\theta [\sin \theta \cos \theta (\cos^2 \phi + \sin^2 \phi - 1)] + 2 r^2 [-\cos \theta \sin \theta \cos \phi \sin \phi + \\
& \cos \theta \sin \theta \sin \phi \cos \phi] =
\end{aligned}$$

$$(dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$$

The final step shows us that the scale factors are :

$$h_r = \sqrt{1}$$

$$h_\theta = \sqrt{r^2} = r$$

$$h_\phi = \sqrt{r^2 \sin^2 \theta} = r \sin \theta$$

b) Unit Vectors

We begin by writing the position vector, \mathbf{r} , as

$$\mathbf{r} = x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}$$

And use the transformation equations to obtain

$$\mathbf{r} = r \sin \theta \cos \phi \hat{\mathbf{x}} + r \sin \theta \sin \phi \hat{\mathbf{y}} + r \cos \theta \hat{\mathbf{z}}$$

We find each unit vector by computing :

$$\hat{\mathbf{e}}_i = \frac{\partial \mathbf{r}}{\partial q_i} / \left| \frac{\partial \mathbf{r}}{\partial q_i} \right|$$

So we have :

$$\hat{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial r} / \left| \frac{\partial \mathbf{r}}{\partial r} \right| = \frac{\sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}}{\sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta}} =$$

$$\sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$\hat{\theta} = \frac{\partial \mathbf{r}}{\partial \theta} / \left| \frac{\partial \mathbf{r}}{\partial \theta} \right| = \frac{r \cos \theta \cos \phi \hat{\mathbf{x}} + r \cos \theta \sin \phi \hat{\mathbf{y}} - r \sin \theta \hat{\mathbf{z}}}{\sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta}} =$$

$$\frac{r \cos \theta \cos \phi \hat{\mathbf{x}} + r \cos \theta \sin \phi \hat{\mathbf{y}} - r \sin \theta \hat{\mathbf{z}}}{r} =$$

$$\cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}$$

$$\hat{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} / \left| \frac{\partial \mathbf{r}}{\partial \phi} \right| = \frac{-r \sin \theta \sin \phi \hat{\mathbf{x}} + r \sin \theta \cos \phi \hat{\mathbf{y}}}{\sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi}} = -\sin \theta \sin \phi \hat{\mathbf{x}} + \sin \theta \cos \phi \hat{\mathbf{y}}$$

c) Cartesian Unit Vectors

Now that we have expressions for the three spherical polar unit vectors in terms of the three Cartesian unit vectors, we can express this as a matrix relationship :

$$\begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \quad (1)$$

Now, since we know the spherical polar coordinate system is orthogonal, the 3 x3 matrix on the left of the equation (1) is orthogonal, so that we also know the inverse of that matrix is equal to its transpose. If we call this matrix M, we can write eq. (1) as :

$$M \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

If we left multiply each side by the inverse of M, we have :

$$M^{-1} M \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = M^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \Rightarrow I \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = M^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = M^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

and we have written the matrix equation to express the Cartesian unit vectors in terms of the spherical polar unit vectors. Now, knowing that for an orthogonal matrix,

$$\mathbf{M}^{-1} = \mathbf{M}^T$$

we have by inspection :

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \mathbf{M}^T \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

Multiplying this matrix equation gives us :

$$\begin{aligned} \hat{\mathbf{x}} &= \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} &= \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} &= \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{aligned}$$

And we now have the Cartesian unit vectors written in terms of the spherical polar unit vectors.

d) Position Vector

We write the position vector as :

$$\mathbf{r} = x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}$$

Substituting the expressions from the transformation equations and for the Cartesian unit vectors, we get :

$$\begin{aligned} \mathbf{r} &= r \sin \theta \cos \phi (\sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}}) + \\ & r \sin \theta \sin \phi (\sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}}) + r \cos \theta (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}) \end{aligned}$$

Multiply through and collect according to unit vector :

$$\begin{aligned} \mathbf{r} &= (r \sin^2 \theta \cos^2 \phi + r \sin^2 \theta \sin^2 \phi + r \cos^2 \theta) \hat{\mathbf{r}} + \\ & (r \sin \theta \cos \theta \cos^2 \phi + r \sin \theta \cos \theta \sin^2 \phi - r \cos \theta \sin \theta) \hat{\boldsymbol{\theta}} + \\ & (-r \sin \theta \cos \phi \sin \phi + r \sin \theta \sin \phi \cos \phi) \hat{\boldsymbol{\phi}} \end{aligned}$$

After some basic trig and algebra, we get the mortifyingly simple :

$$\mathbf{r} = r \hat{\mathbf{r}} \tag{2}$$

We will use this equation to find the expressions for velocity and acceleration in spherical polar coordinates.

d') Velocity

Velocity is the time derivative of position, so we start out easily enough :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}$$

A Quick Excursion to find Time Derivatives of Unit Vectors

Knowing that we have to find an expression for acceleration in the question below, we realize we will need time derivatives for all the unit vectors in spherical polars, so let's take care of that now. We will need to use the expressions for unit vectors in spherical polar coordinates derived earlier; namely:

$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}\end{aligned}$$

When we take the time derivatives of these unit vectors, we have to remember that both θ and ϕ are functions of time, so that we have to employ the product rule for each term. I will first take derivatives with respect to θ , and then will differentiate with respect to ϕ . Finally, I will group terms according to either $\dot{\theta}$ or $\dot{\phi}$.

Starting with the time derivative for the unit vector \mathbf{r} :

$$\begin{aligned}\dot{\hat{\mathbf{r}}} &= \frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}(\cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}) \\ &\quad + \dot{\phi}(-\sin\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}}) \\ &= \dot{\theta}\hat{\boldsymbol{\theta}} + \sin\theta\dot{\boldsymbol{\phi}}\end{aligned}\tag{3}$$

Referring to the definitions of the unit vectors above, we can notice that the parenthetical expression in the top line is just $\hat{\boldsymbol{\theta}}$, and that the parenthetical expression in the lower line is $\sin\theta\hat{\boldsymbol{\phi}}$

Next, we find the equation for $\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}\dot{\hat{\boldsymbol{\theta}}} &= \frac{d\hat{\boldsymbol{\theta}}}{dt} = \dot{\theta}(-\sin\theta\cos\phi\hat{\mathbf{x}} - \sin\theta\sin\phi\hat{\mathbf{y}} - \cos\theta\hat{\mathbf{z}}) \\ &\quad + \dot{\phi}(-\cos\theta\sin\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}}) \\ &= -\dot{\theta}\hat{\mathbf{r}} + \cos\theta\dot{\boldsymbol{\phi}}\end{aligned}\tag{4}$$

Finally, we will find the time derivative for $\hat{\phi}$; once we obtain the derivative, we will need to do a little algebra to express $\dot{\hat{\phi}}$ in terms of \hat{r} and $\hat{\theta}$. First, upon taking the derivative we obtain :

$$\dot{\hat{\phi}} = \frac{d}{dt} \hat{\phi} = \dot{\phi} (-\cos \phi \hat{x} - \sin \phi \hat{y}) = -\dot{\phi} (\cos \phi \hat{x} + \sin \phi \hat{y}) \quad (5)$$

But at this point, we do not see an obvious way to write $\dot{\hat{\phi}}$ in terms of \hat{r} and $\hat{\theta}$. Notice that if we multiply the equation for \hat{r} by $\sin \theta$ and multiply the $\hat{\theta}$ equation by $\cos \theta$, we get:

$$\begin{aligned} \sin \theta \hat{r} &= \sin^2 \theta \cos \phi \hat{x} + \sin^2 \theta \sin \phi \hat{y} - \sin \theta \cos \theta \hat{z} \\ \cos \theta \hat{\theta} &= \cos^2 \theta \cos \phi \hat{x} + \cos^2 \theta \sin \phi \hat{y} + \sin \theta \cos \theta \hat{z} \end{aligned}$$

Adding these two equations gives us :

$$\sin \theta \hat{r} + \cos \theta \hat{\theta} = \cos \phi \hat{x} + \sin \phi \hat{y}$$

Notice that we now have a way to write the right side of equation (5) in terms of the unit vectors \hat{r} and $\hat{\theta}$. And this allows us to finally obtain an expression for $\dot{\hat{\phi}}$:

$$\dot{\hat{\phi}} = -\dot{\phi} (\cos \phi \hat{x} + \sin \phi \hat{y}) = -\dot{\phi} (\sin \theta \hat{r} + \cos \theta \hat{\theta}) \quad (6)$$

Meanwhile, back at the ranch for computing velocity and acceleration ...

We now have expressions for the time derivatives of each unit unit vector, so we can go back to our definition of velocity in spherical coordinates and obtain :

$$\begin{aligned} \mathbf{v} &= \dot{r} \hat{r} + r \dot{\hat{r}} = \dot{r} \hat{r} + r (\dot{\theta} \hat{\theta} + \dot{\phi} \sin \theta \hat{\phi}) = \\ &\dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \dot{\phi} \sin \theta \hat{\phi} \end{aligned} \quad (7)$$

And this is the equation for the velocity of a particle in spherical polar coordinates.

d'') Finding acceleration

Acceleration is the time derivative of velocity, so we can begin our solution to this problem with the deceptively economical equation :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

The fly in this ointment is that equation (7) above gives us an expression for the velocity of a particle in spherical polar coordinates, and we will have to take the time derivative of each term (including unit vectors) to find an expression for acceleration. Using equation (7) for velocity gives us :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}[(\dot{r}\hat{\mathbf{r}}) + (r\dot{\theta}\hat{\boldsymbol{\theta}}) + (r\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}})] = \quad (8)$$

$$\begin{aligned} & (\ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}}) + (\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}}) + \\ & (\dot{r}\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}} + r\ddot{\phi}\sin\theta\hat{\boldsymbol{\phi}} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\boldsymbol{\phi}} + r\dot{\phi}\sin\theta\dot{\hat{\boldsymbol{\phi}}}) \end{aligned} \quad (9)$$

Finally, we use our expressions for time derivatives of unit vectors (eqs. 4 - 6) in eq. (9) and get :

$$\begin{aligned} \mathbf{a} = & (\ddot{r}\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin\theta\hat{\boldsymbol{\phi}})) \\ & + (\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}} + \dot{\phi}\cos\theta\hat{\boldsymbol{\phi}})) \\ & + (\dot{r}\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}} + r\ddot{\phi}\sin\theta\hat{\boldsymbol{\phi}} + \\ & r\dot{\phi}\dot{\theta}\cos\theta\hat{\boldsymbol{\phi}} + r\dot{\phi}\sin\theta(-\dot{\phi}(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}})) \end{aligned} \quad (10)$$

We then multiply out terms and group according to unit vectors, and obtain :

$$\begin{aligned} \mathbf{a} = & (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} \\ & + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{\theta}} \\ & + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\boldsymbol{\phi}} \end{aligned} \quad (11)$$

And we are done. SAVE this assignment; you will need these expressions when you take theoretical mechanics.