PHYS 301
HOMEWORK #10-- SOLUTIONS

1. Solve the PDE

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + y = 0$$

subject to:

$$y(0, t) = y(1, t) = 0$$

Our trial solution will be y = X (x) T (t), when we substitute it into our original differential equation we obtain :

$$X\,T'' - X''\,T + X\,T = 0$$

Dividing through by the solution gives:

$$\frac{T''}{T} - \frac{X''}{X} + 1 = 0$$

Following the hint in the problem, we group terms this way:

$$\frac{T''}{T} + 1 = \frac{X''}{X}$$

b) The left side does not depend on X. Therefore, if I change X the left side will not change. Since this equation must hold true for all X and T, the right side cannot change, thus the right side must equal a constant. The same logic explains why the left side equals a constant.

c) We are told to set

$$\frac{X''}{X} = P$$

If P is a constant, our differential equation gives exponential solutions, which cannot fit the boundary conditions (BCs) (y = 0 at both edges). If P = 0, the solution is a straight line, which can match the boundary conditions only if the line has zero slope, i.e., the trivial solution. If P is negative, the solutions will be sinusoidal, and those can fit the BCs. Thus we have :

$$\frac{X''}{X} + P = -k^2 \Rightarrow X = A\cos kx + B\sin kx$$

d) If y = 0 at x = 0, we know that y (0, t) = A cos 0 + 0 = 0. This tells us A = 0

e) If y (1, t) = 0 ⇒ sin (k) = 0 ⇒ k = n π

f) The T equation becomes :

$$\frac{T''}{T} + 1 = -k^2 = -n^2\pi^2$$

$$\frac{T''}{T} = -\left(1 + n^2\pi^2\right) \Rightarrow T = C\cos\sqrt{n^2\pi^2+1} + D\sin\sqrt{n^2\pi^2+1}$$

g) We are given no initial condition so cannot compute values for the Cs and Ds, the complete solution is then a sum over all possible values of n :

$$y(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(C_n\cos\sqrt{n^2\pi^2+1} + D_n\sin\sqrt{n^2\pi^2+1}\right)$$

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2. Solve the PDE :

$$\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2}$$

subject to :

$$y(0, t) = y(L, t) = 0 \text{ and } y(x, 0) = \sin(\pi x/L)$$

The steps should be familiar now, substitute the trial solution (X(x)T(t)) into the original PDE, divide by the solution, and separate variables, leaving us with:

$$\frac{T'}{T} = c^2\frac{X''}{X}$$

Let' s isolate the X terms, so we have :

$$\frac{1}{c^2}\frac{T'}{T} = \frac{X''}{X} = \text{constant}$$

Because of our BCs, we know that we must have a sinusoidal solution in X, so the constant must be negative, and we have:

$$X = A\cos kx + B\sin kx$$

and the T equation gives us:

$$T = C\,e^{-c^2k^2t}$$

Following a pattern that should be familiar :

$$y(0, t) = 0 \Rightarrow A = 0$$

$$y(L, t) = 0 \Rightarrow k = n\pi/L$$

and our solution becomes the sum over all possible normal modes:

$$y(x, t) = \sum_{n=1}^{\infty} B_n\sin(n\pi x/L)\,e^{-c^2n^2\pi^2t/L}$$

Now using the initial condition:

$$y(x, 0) = \sin(\pi x/L) = \sum_{n=1}^{\infty} B_n\sin(n\pi x/L)$$

And we see that this is just a Fourier sine series. Note carefully though that the function we have to fit is a sine wave, so that we can easily fit this condition with $B_1 = 1$ and all other $B_n = 0$. Our solution then is the single term:

$$y(x, t) = \sin(\pi x / L)\, e^{-c^2 n^2 \pi^2 t/L}$$

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3. We are asked to find the solution for the potential of a sphere of radius a whose surface potential is given by :

$$V_o(\theta) = \sin\theta$$

We are directed to the solutions worked out in the text (and in classnotes). The potential in (or on) a sphere of radius a is given by:

$$V(r, \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

where r is the distance from the center of the sphere and A_m are the coefficients defined as:

$$A_m = \frac{2m+1}{2a^m} \int_0^\pi V_o(\theta)\, P_m(\cos\theta) \sin\theta \, d\theta$$

(this equation is similar to the one used in classnotes if you set x = cos θ and dx = - sin θ dθ).

To make matters simpler without losing any physics, I just set the radius = 1, so for our particular surface potential function we can write:

$$A_m = \frac{2m+1}{2} \int_o^\pi \sin^2\theta\, P_m(\cos\theta)\, d\theta$$

The short *Mathematica* program below will plot out the 20th partial sum of the solution for the potential and superimpose it over the curve sin θ:

In[81]:=
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Clear[a, potential]
a[m_] := a[m] = ((2 m + 1) / 2) Integrate[Sin[θ]^2 LegendreP[m, Cos[θ]], {θ, 0, π}]

potential[θ_] := Sum[a[m] LegendreP[m, Cos[θ]], {m, 0, 20}]

Plot[{Sin[x], potential[x]}, {x, 0, π}, PlotLegends → Automatic]
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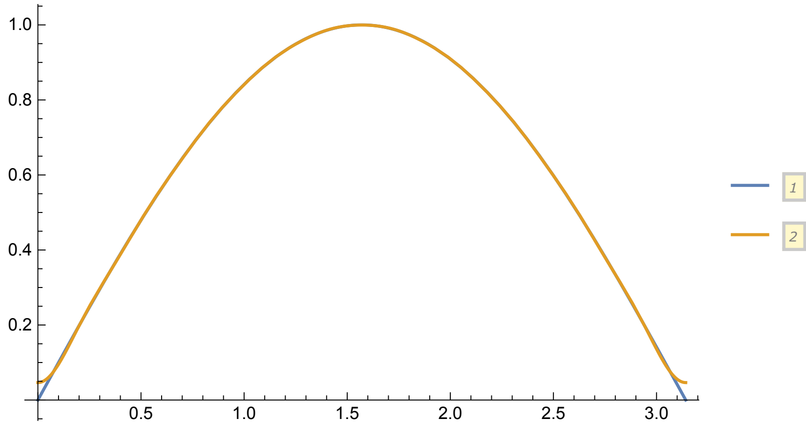
Out[84]=



I' m not sure what error the text was expecting; perhaps forgetting to start the sum at zero, perhaps forgetting to use LegendreP[m, Cos[θ]] rather than LegendreP[m, θ]. Anyway, the above should work.If you look carefully at $\theta = 0$ and π, the computed potential curve deviates slightly from the sin curve.

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4. The problem is identical to the one worked out in detail in the text (pp 581 - 584) except for the BC on the top of the cube. We can thus write in complete generality eq. 11.5 .4 from the text :

$$V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(m\pi x/L) \sin(n\pi y/L) \left(e^{\alpha z/L} - e^{-\alpha z/L}\right)$$

$$\text{where } \alpha = \pi \sqrt{n^2 + m^2}$$

We now use our BC when z = L, and we get:

$$\sin(\pi x/L) + \sin(2\pi y/L) + \sin(2\pi x/L)\sin(\pi y/L) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(m\pi x/L) \sin(n\pi y/L) (e^{\alpha} - e^{-\alpha})$$

As in an earlier problem, our BC consists of sin waves. This means that we can completely specify our solution by setting m=1, n =2 and also m=2, n=1. This gives us a value of A_{mn}:

$$A_{mn}\left(e^{\sqrt{5}\,\pi} - e^{-\sqrt{5}\,\pi}\right) = 1 \Rightarrow A_{mn} = \frac{1}{e^{\sqrt{5}\,\pi} - e^{-\sqrt{5}\,\pi}}$$

Since there are only two non-zero terms (m=1,n=2 and m=2,n=1), our total solution is:

$$V(x, y, z) = \frac{e^{\sqrt{5}\,\pi z/L} - e^{-\sqrt{5}\,\pi z/L}}{e^{\sqrt{5}\,\pi} - e^{-\sqrt{5}\,\pi}}[\sin(\pi x/L)\sin(2\pi y/L) + \sin(2\pi x/L)\sin(\pi y/L)]$$