PHYS 301

HOMEWORK #2--Solutions

Question #1

The transformation equations from Cartesian to spherical polar coordinates are :

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

a)**Scale Factors**

We begin by constructing an expression for $(ds)^2$ by computing the values of dx, dy and dz:

$$dx = \sin\theta \cos\phi \, dr + r\cos\theta\cos\phi \, d\theta - r\sin\theta\sin\phi \, d\phi$$

$$dy = \sin\theta\sin\phi \, dr + r\cos\theta\sin\phi \, d\theta + r\sin\theta\cos\phi \, d\phi$$

$$dz = \cos\theta \, dr - r\sin\theta \, d\theta$$

We square each of these terms; I will write these expressions with the perfect squares first followed by the cross - terms :

$$(dx)^2 = \sin^2\theta\cos^2\phi\,(dr)^2 + r^2\cos^2\theta\cos^2\phi\,(d\theta)^2 + r^2\sin^2\theta\sin^2\phi\,(d\phi)^2$$
$$+ 2\,r\sin\theta\cos\theta\cos^2\phi\,dr\,d\theta \;\; - 2\,r\sin^2\theta\cos\phi\sin\phi\,dr\,d\phi - 2\,r^2\cos\theta\sin\theta\cos\phi\sin\phi\,d\theta\,d\phi$$

$$(dy)^2 = \sin^2\theta\sin^2\phi\,(dr)^2 + r^2\cos^2\theta\sin^2\phi\,(d\theta)^2 + r^2\sin^2\theta\cos^2\phi\,(d\phi)^2 +$$
$$2\,r\sin\theta\cos\theta\sin^2\phi\,dr\,d\theta + 2\,r\sin^2\theta\sin\phi\cos\phi\,dr\,d\phi + 2\,r^2\cos\theta\sin\theta\sin\phi\cos\phi\,d\theta\,d\phi$$

$$(dz)^2 = \cos^2\theta\,(dr)^2 + r^2\sin^2\theta\,(d\theta)^2 - 2\,r\cos\theta\sin\theta\,dr\,d\theta.$$

Now, we add each of these expressions, and group like terms to obtain :

$$(dx)^2 + (dy)^2 + (dz)^2 =$$

$$(dr)^2\left[\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta\right]+$$
$$r^2(d\theta)^2\left[\cos^2\theta\cos^2\phi+\sin^2\theta\cos^2\phi+\sin^2\theta\right]+r^2(d\phi)^2\left[\sin^2\theta\sin^2\phi+\sin^2\theta\cos^2\phi\right]+$$
$$dr\,d\theta\left[2\,r\sin\theta\cos\theta\cos^2\phi+2\,r\sin\theta\cos\theta\sin^2\phi-2\,r\cos\theta\sin\theta\right]+$$
$$dr\,d\phi\left[-2\,r\sin^2\theta\cos\phi\sin\phi\,dr\,d\phi+2\,r\sin^2\theta\sin\phi\cos\phi\,dr\,d\phi\right]+$$
$$d\theta d\phi\left[-2\,r^2\cos\theta\sin\theta\cos\phi\sin\phi\,d\theta\,d\phi+2\,r^2\cos\theta\sin\theta\sin\phi\cos\phi\,d\theta\,d\phi\right]=$$

$$(dr)^2\left[\sin^2\theta+\cos^2\theta\right]+r^2(d\theta)^2\left[\cos^2\theta+\sin^2\theta\right]+r^2(d\phi)^2\left[\sin^2\theta\left(\sin^2\phi+\cos^2\phi\right)\right]-$$
$$2\,r dr\,d\theta\left[\sin\theta\cos\theta\left(\cos^2\phi+\sin^2\phi-1\right)\right]+2\,r^2[-\cos\theta\sin\theta\cos\phi\sin\phi+$$
$$\cos\theta\sin\theta\sin\phi\cos\phi]=$$

$$(dr)^2+r^2(d\theta)^2+r^2\sin^2\theta\,(d\phi)^2$$

The final step shows us that the scale factors are :

$$h_r = \sqrt{1}$$

$$h_\theta = \sqrt{r^2} = r$$

$$h_\phi = \sqrt{r^2\sin^2\theta} = r\sin\theta$$

b) Unit Vectors

We begin by writing the position vector, **r**, as

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

And use the transformation equations to obtain

$$\mathbf{r} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$$

We find each unit vector by computing :

$$\hat{\mathbf{e}}_{\mathbf{i}} = \frac{\partial\mathbf{r}}{\partial q_i}\Big/\left|\frac{\partial\mathbf{r}}{\partial q_i}\right|$$

So we have :

$$\hat{\mathbf{r}} = \frac{\partial\mathbf{r}}{\partial r}\Big/\left|\frac{\partial\mathbf{r}}{\partial r}\right| = \frac{\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}}{\sqrt{\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta}} =$$

$$\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \frac{\partial \mathbf{r}}{\partial \theta} \Big/ \left| \frac{\partial \mathbf{r}}{\partial \theta} \right| = \frac{\mathrm{r} \cos\theta \cos\phi\, \hat{\mathbf{x}} + \mathrm{r}\cos\theta \sin\phi\, \hat{\mathbf{y}} - \mathrm{r}\sin\theta\, \hat{\mathbf{z}}}{\sqrt{\mathrm{r}^2 \cos^2\theta \cos^2\phi + \mathrm{r}^2\cos^2\theta\sin^2\phi + \mathrm{r}^2\sin^2\theta}} =$$

$$\frac{\mathrm{r} \cos\theta \cos\phi\, \hat{\mathbf{x}} + \mathrm{r}\cos\theta \sin\phi\, \hat{\mathbf{y}} - \mathrm{r}\sin\theta\, \hat{\mathbf{z}}}{\mathrm{r}} =$$

$$\cos\theta \cos\phi\, \hat{\mathbf{x}} + \cos\theta \sin\phi\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left| \frac{\partial \mathbf{r}}{\partial \phi} \right| = \frac{-\mathrm{r} \sin\theta \sin\phi\, \hat{\mathbf{x}} + \mathrm{r}\sin\theta \cos\phi\, \hat{\mathbf{y}}}{\sqrt{\mathrm{r}^2 \sin^2\theta \sin^2\phi + \mathrm{r}^2\sin^2\theta\cos^2\phi}} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}$$

c) Cartesian Unit Vectors

Now that we have expressions for the three spherical polar unit vectors in terms of the three Cartesian unit vectors, we can express this as a matrix relationship :

$$\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} \qquad (1)$$

Now, since we know the spherical polar coordinate system is orthogonal, the 3 x3 matrix on the left of the equation (1) is orthogonal, so that we also know the inverse of that matrix is equal to its transpose. If we call this matrix M, we can write eq. (1) as :

$$\mathrm{M} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

If we left multiply each side by the inverse of M, we have :

$$\mathrm{M}^{-1}\,\mathrm{M} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \mathrm{M}^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} \Rightarrow \mathrm{I} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \mathrm{M}^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \mathrm{M}^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

and we have written the matrix equation to express the Cartesian unit vectors in terms of the spherical polar unit vectors. Now, knowing that for an orthogonal matrix,

$$M^{-1} = M^{T}$$

we have by inspection :

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = M^{T} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

Multiplying this matrix equation gives us :

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

And we now have the Cartesian unit vectors written in terms of the spherical polar unit vectors.

d) Position Vector

We write the position vector as :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

Substituting the expressions from the transformation equations and for the Cartesian unit vectors, we get :

$$\mathbf{r} = r\sin\theta\cos\phi\left(\sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}\right) + \\ r\sin\theta\sin\phi\left(\sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}\right) + r\cos\theta\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Multiply through and collect according to unit vector :

$$\mathbf{r} = \left(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi + r\cos^2\theta\right)\hat{\mathbf{r}} + \\ \left(r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\cos\theta\sin\theta\right)\hat{\boldsymbol{\theta}} + \\ \left(-r\sin\theta\cos\phi\sin\phi + r\sin\theta\sin\phi\cos\phi\right)\hat{\boldsymbol{\phi}}$$

After some basic trig and algebra, we get the mortifyingly simple :

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad (2)$$

We will use this equation to find the expressions for velocity and acceleration in spherical polar coordinates.