2. ***Solution***: We need to convert all terms, both the scalar and unit vector, into Cartesian coordinates. To convert the scalar term, 1/r, recall that r is the distance from the origin so that :

$$r = \sqrt{x^2 + y^2 + z^2}$$

We use the transformation equations and results from the last homework to write the unit vector r:

$$x = r \sin\theta \cos\theta \quad (1)$$

$$y = r \sin\theta \sin\phi \quad (2)$$

$$z = r \cos\theta \quad (3)$$

and :

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

If we divide each of the transformation equations (eqs. 1-3) by r, we see that:

$$\frac{x}{r} = \sin\theta\cos\theta; \quad \frac{y}{r} = \sin\theta\sin\phi; \quad \frac{z}{r} = \cos\theta$$

We can use these results to rewrite $\hat{r}$:

$$\hat{\mathbf{r}} = \frac{1}{r}\left(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\right)$$

so that:

$$\frac{1}{r}\hat{\mathbf{r}} = \frac{1}{r}\cdot\frac{1}{r}\left(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\right) = \frac{1}{x^2+y^2+z^2}\left(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\right)$$

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3. ***Solution:*** Here we are asked to find the work done by a force over a specific contour. This means we want to evaluate the line integral :

$$W = \oint \mathbf{F}\cdot d\mathbf{l}$$

where we need to remember to convert both the force and line element to polar coordiantes.

The function will transform as:

$$\mathbf{f} = 2(\rho\sin\phi)\left(\cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}}\right) - (\rho\cos\phi)\left(\sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}}\right) =$$

$$\rho\sin\phi\cos\phi\,\hat{\boldsymbol{\rho}} - \rho\left(2\sin^2\phi + \cos^2\phi\right)\hat{\boldsymbol{\phi}}$$

and the line element becomes:

$$d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}}$$

The line integral becomes:

$$W = \int_0^{\pi}\left[\rho\sin\phi\cos\phi\,\hat{\boldsymbol{\rho}} - \rho\left(2\sin^2\phi + \cos^2\phi\right)\hat{\boldsymbol{\phi}}\right]\cdot\left(d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}}\right)$$

$$= \int_0^{\pi}\rho\,(\sin\phi\cos\phi)\,d\rho - \rho^2\left(1+\sin^2\phi\right)d\phi$$

Since our contour is the circle $\rho = 2$, $d\rho = 0$ (since ρ is a constant along the contour), and our integral becomes simply:

$$W = -4 \int_0^{\pi} \left(1 + \sin^2\phi\right) d\phi = -6\pi$$

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4. ***Solution:*** We begin with the transformation equations :

$$x = a \cosh u \cos v$$

$$y = a \sinh u \sin v$$

z = z

Taking differentials :

$$dx = a(\sinh u \cos v\, du - \cosh u \sin v\, dv)$$

$$dy = a(\cosh u \sin v\, du + \sinh u \cos v\, dv)$$

$$dz = dz$$

Squaring and adding:

$$ds^2 =$$

$$dx^2 + dy^2 + dz^2 = a^2\left(\sinh^2 u \cos^2 v\,(du)^2 - 2 \sinh u \cosh u \cos v \sin v\, du\, dv + \cosh^2 u \sin^2 v\,(dv)^2\right)$$

$$+ a^2\left(\cosh^2 u \sin^2 v\,(du)^2 + 2 \cosh u \sin v \sinh u \cos v\, du\, dv + \sinh^2 u \cos^2 v\,(dv)^2\right)$$

$$+(dz)^2$$

Note that all the mixed derivative terms cancel, indicating that this is in fact an orthogonal transformation.

Summing terms and grouping, we get:

$$(ds)^2 = a^2\left(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v\right)(du)^2$$

$$+ a^2\left(\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v\right)(dv)^2 + (dz)^2$$

The scale factor for z is trivially 1. We can simplify the parenthetical expressions by recalling:

$$\cosh^2 x = 1 + \sinh^2 x$$

and obtain for h_u:

$$a^2\left(\sinh^2 u \cos^2 v + \left(1 + \sinh^2 u\right)\sin^2 v\right)(du)^2$$

$$= \left(\sinh^2 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 v\right)(du)^2$$

$$\Rightarrow h_u = a\sqrt{\sinh^2 u + \sin^2 v}$$

Using the same identity, you will find that $h_v = h_u$.

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5. ***Solution:*** We begin with our original equation :

$$\ddot{r} - \frac{h^2}{r^3} = \frac{-GM}{r^2}$$

and our goal is to transform this from an equation in terms of r(t) to an equation in terms of u(θ) where u = 1/r. We will need to make use of the chain rule and also the result:

$$r^2\,\dot{\theta} = h \Rightarrow \dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2} = h\,u^2$$

Let' s focus on the most complex of these terms, the second derivative. Our goal is to convert :

$$\frac{d^2 r}{dt^2} \rightarrow \frac{d^2 u}{d\theta^2}$$

we will need to start with the first derivative term. We can use the chain rule to write:

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\left(h\,u^2\right) = -h\frac{du}{d\theta} \equiv w$$

To find the second derivative, we write:

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{dw}{dt} = \frac{dw}{d\theta}\frac{d\theta}{dt} = -h\frac{d^2 u}{d\theta^2}\cdot h\,u^2$$

So the original differential equation becomes:

$$-h^2\,u^2\frac{d^2 u}{d\theta^2} - h^2\,u^3 = -GM\,u^2$$

Divide through by $-h^2\,u^2$ and we obtain:

$$\frac{d^2 u}{d\theta^2} + u = \frac{GM}{h^2}$$