PHYS 301
HOMEWORK #4

Due : 13 February 2017

Do all integrals by hand; you may check your answers via *Mathematica* but must show all work.

1. Show for m, n integers :

$$\int_{-\pi}^{\pi} \sin(m\,x)\sin(n\,x)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(n\,x)\cos(m\,x)\,dx = 0$$

$$\int_{-\pi}^{\pi} \cos(m\,x)\cos(n\,x)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \\ 2\pi, & m = n = 0 \end{cases}$$

Solution : We will rewrite each integrand using the sin and cos addition formulae :

$\sin(m \pm n)\,x = \sin(m\,x)\cos(n\,x) \pm \sin(n\,x)\cos(m\,x)$

$\cos(m \pm n)\,x = \cos(m\,x)\cos(n\,x) \mp \sin(m\,x)\sin(n\,x)$

a) Adding the sin addition/subtraction formulae gives:

$$\sin(m+n)\,x + \sin(m-n)\,x = 2\sin(m\,x)\cos(n\,x) \tag{1}$$

This allows us to write the second integral above as:

$$\int_{-\pi}^{\pi} \sin(n\,x)\cos(m\,x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)\,x\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(m-n)\,x\,dx \tag{2}$$

These integrals return cos(p x) where p is an integer evaluated at π and $-\pi$. Since cos is an even function, each integral evaluates to zero when m≠ n. In the case where m = n, the integrals become:.

$$\int_{-\pi}^{\pi} \sin(2\,m\,x)\,dx \quad \text{and} \int 0\,dx$$

both of which are easily (or trivially) shown to be zero.

b) If we add the cos addition/subtraction formulae, we get:

$$\int_{-\pi}^{\pi} \cos(m\,x)\cos(n\,x)\,dx = \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos(m+n)\,x + \cos(m-n)\,x\,dx\right]$$

These integrals return sin(p x) (where p is an integer) evaluted at π and $-\pi$; since sin is zero at those values, the integral is zero for all m≠n. If m = n, the integral becomes

$$\int_{-\pi}^{\pi} \cos^2(m\,x)\,dx = \int_{-\pi}^{\pi}\left(\frac{1+\cos(2\,x))}{2}\right)dx = \pi$$

(we use the trig identities cos(2x) = $\cos^2$x - $\sin^2$x and $\sin^2$x = 1 - $\cos^2$x)

If m = n = 0, cos (m x) = cos (n x) = 1, and the integral of 1 on this interval is 2 π.

c) We subtract the subtraction/addition cos formulae and get :

$$\int_{-\pi}^{\pi} \sin(m\,x)\sin(n\,x)\,dx = \frac{1}{2}\left[\int_{-\pi}^{\pi}(\cos(m-n)\,x - \cos(m+n)\,x)\,dx\right]$$

Using prior reasining, each integral on the right returns sin(p x) on [-π,π], so all these terms are zero for m≠n. If m = n ≠ 0, we have:

$$\int_{-\pi}^{\pi} \sin^2(m\,x)\,dx = \pi$$

If m = n = 0, the integral is zero since sin (0) = 0.

For the remaining problems we will use these definitions of the Fourier series:

$$a_o = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,dx$$

and for the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\,x) + \sum_{n=1}^{\infty} b_n \sin(n\,x)$$

2. Consider the function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Find the Fourier coefficients and then write the Fourier series both in closed form and by writing out the first three non zero terms of each series.

Solution : We find the Fourier coefficients, using integration by parts where needed :

$$a_o = \frac{1}{2\,\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(n\,x)\,dx = \frac{1}{\pi\,n}\,x\sin(n\,x)\Big|_0^{\pi} - \frac{1}{\pi\,n}\int_0^{\pi}\sin(n\,x)\,dx = \frac{1}{\pi\,n^2}\cos(n\,x)\Big|_0^{\pi}$$

$$= \frac{1}{\pi\,n^2}(\cos(n\,\pi) - 1) = \begin{cases} 0, & \text{n even} \\ -2/n^2\,\pi, & \text{n odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(n\,x) = \frac{1}{\pi}\left[\frac{-1}{n}\,x\cos(n\,x)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(n\,x)\,dx\right]$$

The last integral goes to zero since it returns sin(n x), so we are left with:

$$b_n = \frac{-1}{n\pi}(\pi\cos(n\pi) - 0) = \frac{-1}{n}(-1)^n$$

The Fourier series is:

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=odd}^{\infty}\frac{\cos(n x)}{n^2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n x)}{n}$$

The first three terms yield:

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos(3x)}{9} + \frac{\cos(5x)}{25} + ...\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin(3x)}{3} - ...\right)$$

Plotting two cycles and verifying with *Mathematica*:

In[22]:=
```
g1 = Plot[π / 4 - (2 / π) Sum[Cos[n x] / n^2, {n, 1, 31, 2}] +
   Sum[(-1) ^ (n + 1) Sin[n x] / n, {n, 1, 31}], {x, -π, 3 π}]
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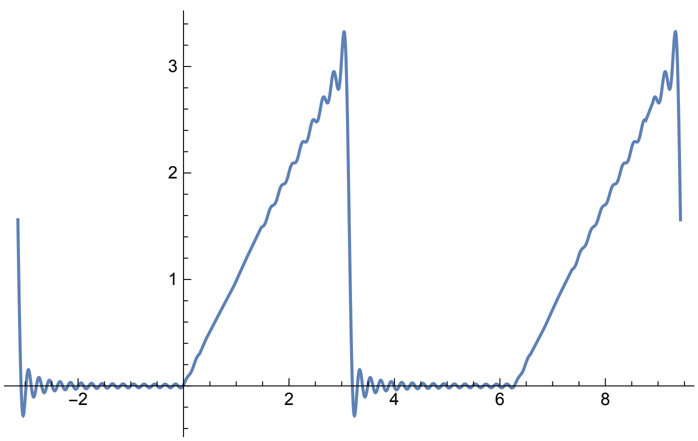
Out[22]=



Just to be sure, we overlay the line y = x in red :

In[25]:=
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g2 = Plot[x, {x, 0, π}, PlotStyle → Red];
Show[g1, g2]
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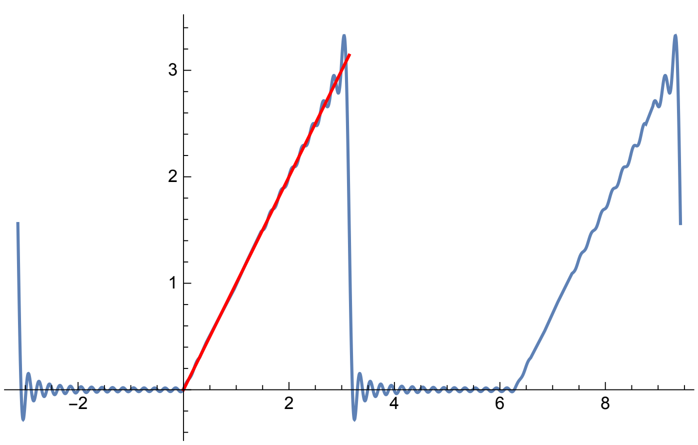
Out[26]=



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3. Consider the function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$$

Find the Fourier coefficients and write the Fourier series both in closed form and by writing out the first three non-zero terms of each series.

Solution : We find the Fourier coefficients by making use of integration by parts where needed.

$$a_o = \frac{1}{2\pi}\int_o^{\pi} x^2\, dx = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos(n x)\, dx = \frac{1}{\pi}\left[\frac{1}{n} x^2 \sin(n x)\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin(n x)\, dx\right]$$

$$= \frac{1}{\pi}\left[0 - \frac{2}{n}\left(\frac{-1}{n} x \cos(n x)\Big|_0^{\pi}\right) + \frac{2}{n^2}\int_0^{\pi} \cos(n x)\, dx\right]$$

The final integral above is zero since it returns sin(n x) to be evaluated at 0 and π, so the a_n coefficients are:

$$a_n = \frac{1}{\pi}\left(\frac{2}{n^2}\pi \cos(n\pi) - 0\right) = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin(n x)\, dx = \frac{1}{\pi}\left[\frac{-1}{n} x^2 \cos(n x)\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x \cos(n x)\, dx\right]$$

$$= \frac{-\pi}{n}\cos(n\pi) + \frac{2}{n\pi}\int_0^{\pi} x \cos(n x)\, dx = \frac{-\pi}{n}(-1)^n + \frac{2}{n\pi}\left[\frac{1}{n} x \sin(n x)\Big|_0^{\pi} - \right.$$

$$\left.\frac{1}{n}\int_0^{\pi} \sin(n x)\, dx\right] = \frac{-\pi}{n}(-1)^n - \frac{2}{n^3\pi}\cos(n x)\Big|_0^{\pi}\Big]$$

$$= \frac{-\pi}{n}(-1)^n + \frac{2}{n^3\pi}(1 - (-1)^n)$$

or :

$$b_n = \begin{cases} -\pi/n, & \text{n even} \\ -\left(4 - n^2\pi^2\right)/n^3\pi, & \text{n odd} \end{cases}$$

The Fourier series can be written:

$$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \cos(n x)}{n^2} - \pi\sum_{n=even}^{\infty}\frac{\sin(n x)}{n} + \sum_{n=od}^{\square}\frac{\left(4 - n^2\pi^2 \sin(n x)\right.}{n^3}$$

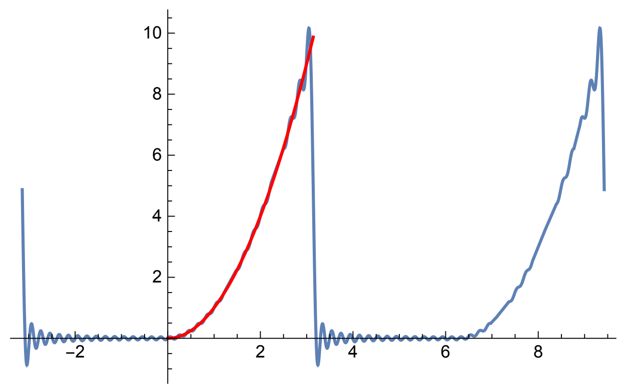
or :

$$f(x) = \frac{\pi^2}{6} - 2\left(\cos x - \frac{\cos(2x)}{4} + \frac{\cos(3x)}{9} -,,,\right) - \pi\left(\frac{\sin(2x)}{2} + \frac{\sin(4x)}{4} + \ldots\right) +$$

$$\left(\left(4 - \pi^2\right)\sin x + \frac{\left(4 - 9\pi^2\right)\sin(3x)}{27} + \frac{\left(4 - 25\pi^2\right)\sin(5x)}{125} + \ldots\right)$$

Verifying with *Mathematica*:

```
Clear[a0, a, bodd, beven]
a0 = π^2 / 6;
a[n_] := 2 (-1)^n / n^2
beven[n_] := -π / n
bodd[n_] := - (4 - n^2 π^2) / (n^3 π)
g1 = Plot[a0 + Sum[a[n] Cos[n x], {n, 1, 31}] + Sum[beven[n] Sin[n x], {n, 2, 30, 2}] +
    Sum[bodd[n] Sin[n x], {n, 1, 31, 2}], {x, -π, 3 π}];
g2 = Plot[x^2, {x, 0, π}, PlotStyle → Red];
Show[g1, g2]
```



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4. Find the Fourier coefficients and write the Fourier series in closed form and also the first three non - zero terms of each series for :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin(2x), & 0 < x < \pi \end{cases}$$

Solution : The danger here is to assume that orthogonality will apply since f (x) = sin (2 x). However, since the limits of integration are [0, π] and not [-π, π], we must do the evaluations explicitly.

$$a_o = \frac{1}{2\pi}\int_0^\pi \sin(2x)\,dx = \frac{1}{2\pi}\left(\frac{-1}{2}\cos(2x)\Big|_0^\pi\right) = 0$$

We use eqs. (1 and 2) from problem one to help with the evaluation of a_n and b_n.

$$a_n = \frac{1}{\pi}\int_0^\pi \sin(2x)\cos(nx)\,dx = \frac{1}{2\pi}\left[\int_0^\pi \sin(n+2)x - \sin(n-2)x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(-\frac{\cos(n+2)x}{n+2} + \frac{\cos(n-2)x}{n-2}\right)\Big|_0^\pi\right] = \begin{cases} 0, & \text{n even} \\ -4/\pi\left(n^2-4\right), & \text{n odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\sin(nx)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi}(\cos(n-2)x - \cos(n+2)x)\,dx\right]$$
$$= \frac{1}{2\pi}\left[\left(\frac{\sin(n-2)x}{n-2} - \frac{\sin(n+2)x}{n+2}\right)\Big|_0^{\pi}\right]$$

It is common for students to get to this point and conclude that all the b_n values are zero since our integrals return sin(n x) where n is an integer on $[0,\pi]$. And this is true for all values of n except n =2. In this case, our integral becomes:

$$b_2 = \frac{1}{\pi}\int_0^{\pi} \sin^2(2x)\,dx = \frac{1}{2}$$

and our Fourier series is:

$$f(x) = \frac{\text{Sin}(2x)}{2} - 4\sum_{n=\text{odd}}^{\infty}\frac{\cos(nx)}{n^2-4}$$
$$= \frac{\sin(2x)}{2} + 4\left(\frac{\cos x}{3} - \frac{\cos 3x}{5} - \frac{\cos 5x}{21} - \ldots\right)$$

Verifying with *Mathematica*:

```
In[147]:= Plot[Sin[2 x] / 2 - (4 / π) Sum[Cos[n x] / (n^2 - 4), {n, 1, 31, 2}], {x, -π, π}]
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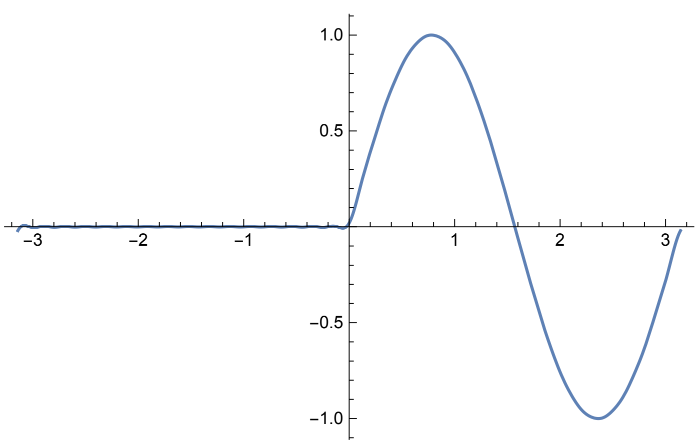
Out[147]=


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5. Find the Fourier coefficients and Fourier series for the function :

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Write out the first three non-zero terms of each series.

Solution :

This is a relief after all the integration by parts we've just done. And if we look a litltle more closely, our problem can be simplified even more. Note that f is an odd function, this means that the only non-zero coefficients will be the b_n terms. Moreover since the function is odd, we know that:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(n x)\,dx$$

In our case, this becomes:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi}\sin(n x)\,dx = \frac{-2}{\pi n}\cos(n x)\Big|_0^{\pi} = \frac{2}{\pi n}(1-(-1)^n) = \begin{cases} 4/\pi n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

and we have simply:

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$$

In[148]:=
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Plot[(4 / π) Sum[Sin[n x] / n, {n, 1, 31, 2}], {x, −3 π, 3 π}]
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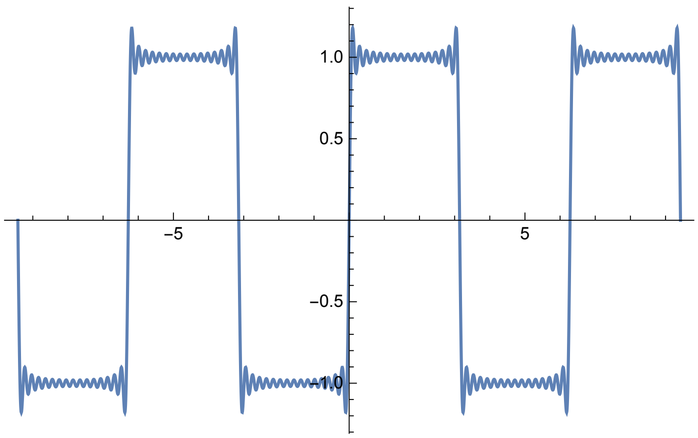
Out[148]=



Now, is the problem even simpler than this. Could we have used another Fourier series that we have computed and deduced this one without ever taking an integral? (Hint : compare this series with the first example in class; f (x) = 1 for $0 < x < \pi$ and f (x) = 0 for $-\pi < x < 0$), see how they compare.)