PHYS 301
HOMEWORK #5-- SOLUTIONS

1. We are asked to find the complex Fourier series for the function :

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

We find complex coefficients from:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

In our case, this becomes:

$$c_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -e^{-inx}\, dx + \int_{0}^{\pi} e^{-inx}\, dx\right]$$

$$= \frac{1}{2\pi}\left[-\left(\frac{-1}{in}\right) e^{-inx}\Big|_{-\pi}^{0} + \left(\frac{-1}{in}\right) e^{-inx}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{in}\left(1 - e^{-in\pi}\right) - \frac{1}{in}\left(e^{-in\pi} - 1\right)\right]$$

$$= \frac{1}{2\pi i n}\left[2\left(1 - e^{-in\pi}\right)\right] = \frac{1}{\pi i n}\left[1 - e^{-in\pi}\right]$$

We know that $e^{-in\pi} = (-1)^n$, so our coefficients become:

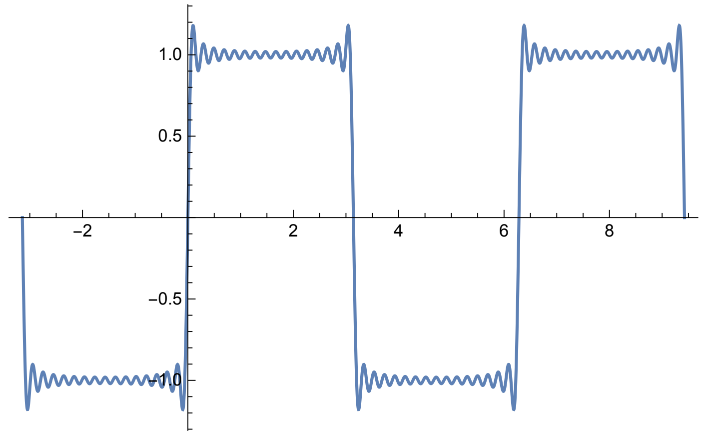
$$c_n = \begin{cases} 0, & n \text{ even} \\ 2/\pi i n, & n \text{ odd} \end{cases}$$

Since the function is odd, we can see that $c_o = 0$. Our Fourier series is:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{2}{\pi}\left[\frac{\left(e^{ix} - e^{-ix}\right)}{i} + \frac{e^{3ix} - e^{-3ix}}{3i} + \ldots\right] = \frac{4}{\pi}\sum_{\text{odd}}^{\infty} \frac{\sin(nx)}{n}$$

Verifying with *Mathematica*:

In[13]:= `Plot[(4 / π) Sum[Sin[n x] / n, {n, 1, 31, 2}], {x, -π, 3 π}]`

Out[13]=


2. In this case, our function is

$$V(t) = t/\pi, \quad -\pi < t < \pi$$

(Note that we must divide by π since the amplitude of the wave is 1 (and not π). Our Fourier coefficients are found via:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = \frac{1}{2\pi^2}\int_{-\pi}^{\pi} t\, e^{-int}\, dt$$

Integration by parts yields:

$$c_n = \frac{1}{2\pi^2}\left[\frac{-1}{in} t\, e^{-int}\Big|_{-\pi}^{\pi} - \left(\frac{-1}{in}\right)\int_{-\pi}^{\pi} e^{-int}\, dt\right]$$

$$= \frac{1}{2\pi^2}\left[\frac{-1}{in}\left(\pi\, e^{-in\pi} - (-\pi)\, e^{-in(-\pi)}\right) + \frac{1}{i^2 n^2}\left(e^{-in\pi} - e^{in\pi}\right)\right]$$

The last term on the right is zero, so I can rewrite the coefficients as:

$$c_n = \frac{-1}{2\pi^2 in}\left(2\pi(-1)^n\right) = \frac{-(-1)^n}{\pi i n}$$

This definition of c_n can clearly not hold for n = 0, but since the function is odd on [-π,π] we know $c_o = 0$.

The Fourier series is:

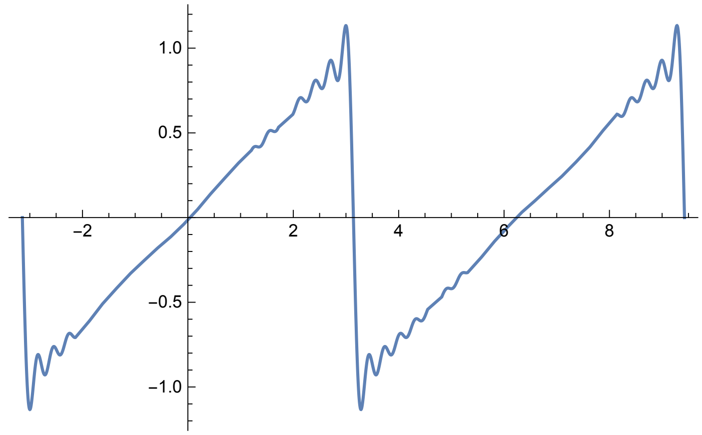
$$V(t) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{1}{\pi}\left[\frac{e^{ix} - e^{-ix}}{i} - \frac{\left(e^{2ix} - e^{-2ix}\right)}{2i} + \frac{\left(e^{3ix} - e^{3ix}\right)}{3i} - \right]\ldots$$

$$= \frac{2}{\pi}\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right]$$

Verifying through *Mathematica*:

In[17]:=
```
Plot[(2 / π) Sum[-(-1)^n Sin[n x] / n, {n, 1, 21}], {x, -π, 3 π}]
```

Out[17]=



3. Now, we have (almost) the same function, but on a different domain. We have :

$$f(x) = x, \; 0 < x < 5$$

If our function is 2L periodic on [0,5], L = 5/2, and our equations for the coefficients and Fourier series are:

$$c_n = \frac{1}{2(5/2)} \int_0^5 f(x)\, e^{-2 i n \pi x/5}\, dx = \frac{1}{5} \int_0^5 f(x)\, e^{-2 i n \pi x/5}\, dx$$

and

$$f(x) = \Sigma c_n e^{2 i n \pi x/5}$$

We' ve already integrated this function by parts (problem 2), so let' s make our lives easy :

Simplify[Integrate[x Exp[−2 I n π x / 5] / 5, {x, 0, 5}], Assumptions −> n ∈ Integers]

Out[18]= $\frac{5\,i}{2\,n\,\pi}$

Remembering that i = -1/i, this becomes:

$$c_n = \frac{-5}{2 \pi i n}$$

Now, this clearly will not give us a meaningful expression for n = 0, so we compute directly :

$$c_0 = \frac{1}{5} \int_0^5 x\, dx = \frac{5}{2}$$

which we could have guessed since that is the average value of x on [0,5]

Our Fourier series is:

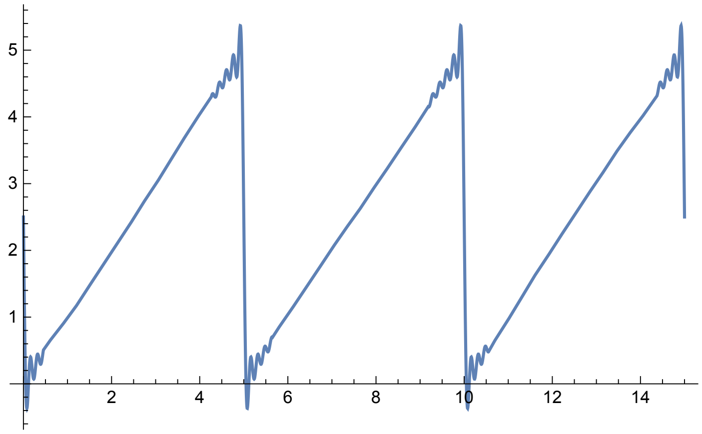
$$f(x) = \frac{5}{2} - \frac{5}{2\pi} \Sigma \frac{e^{2 \pi i n x/5}}{i n}$$

$$= \frac{5}{2} - \frac{5}{2\pi}\left(2 \sin(2 \pi x/5) + \frac{2 \sin(4 \pi x/5)}{2} + 2\, \frac{\sin(6 \pi x/5)}{3} + \ldots\right)$$

Verifying with *Mathematica*:

In[27]:=
```
Plot[5/2 - 5/(π) Sum[Sin[2 n π x/5]/n, {n, 1, 31}], {x, 0, 15}]
```

Out[27]=



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4. a) We are asked to consider the function :

$$f(x) = \sin x + \sin 3x + \sin 10x$$

We know that sin x is 2π periodic. When x = 2π, sin 3x is sin 6π and sin 10x becomes sin 20π, all of which have the same value. Therefore, the function f(x) is also 2π period.

Since the function is 2π periodic, $f(\pi/6) = f(13\pi/6) = f(25\pi/6)$, which is easily shown via:

In[35]:=
```
f[x_] := Sin[x] + Sin[3 x] + Sin[10 x]
Print[f[π/6], "    ", f[13 π/6], "    ", f[25 π/6]]
```

$\frac{3}{2} - \frac{\sqrt{3}}{2} \quad \frac{3}{2} - \frac{\sqrt{3}}{2} \quad \frac{3}{2} - \frac{\sqrt{3}}{2}$

f(x) will have the same value for all x that satisfy $f(x) = f(x + 12\pi/6)$, so $f(37\pi/6)$, $f(49\pi/6)$, $f(61\pi/6)$ etc all yield the same result:

In[37]:=
```
Print[f[37 π/6], "    ", f[49 π/6], "    ", f[61 π/6]]
```

$\frac{3}{2} - \frac{\sqrt{3}}{2} \quad \frac{3}{2} - \frac{\sqrt{3}}{2} \quad \frac{3}{2} - \frac{\sqrt{3}}{2}$

b) Let g (x) = g (x + 5)

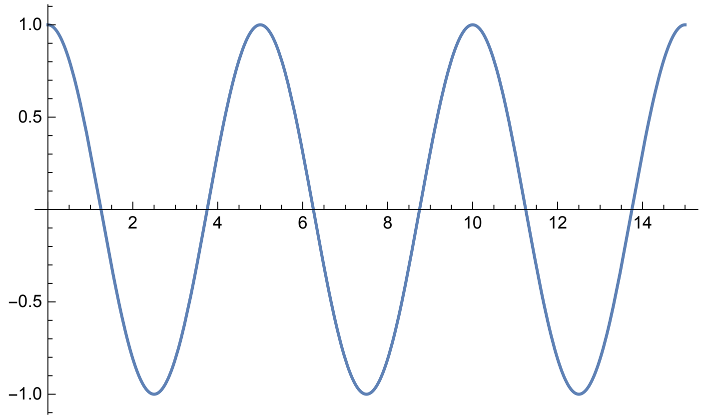
The period of a sin or cos function is related to the coefficient p in cos(p x) by

$$\text{period} = 2\pi/p$$

therefore, if the period is 5, $p = 2\pi/5$, as shown by :

In[38]:= `Plot[Cos[2 π x / 5], {x, 0, 15}]`

Out[38]=

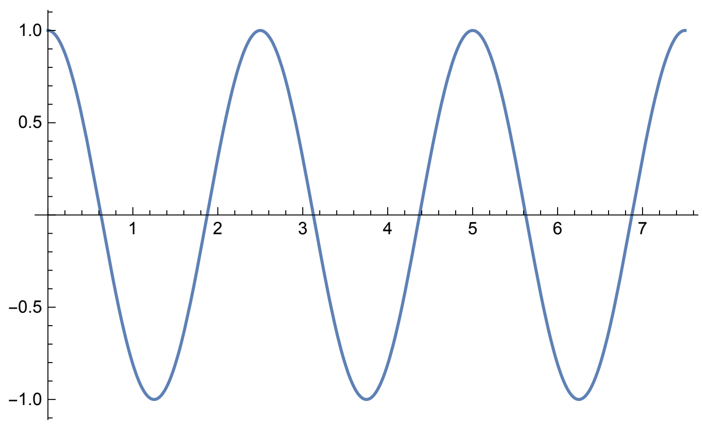


and we get three full cycles.

If the period is 5/2, then p = 4 π/5:

In[39]:= `Plot[Cos[4 π x / 5], {x, 0, 7.5}]`

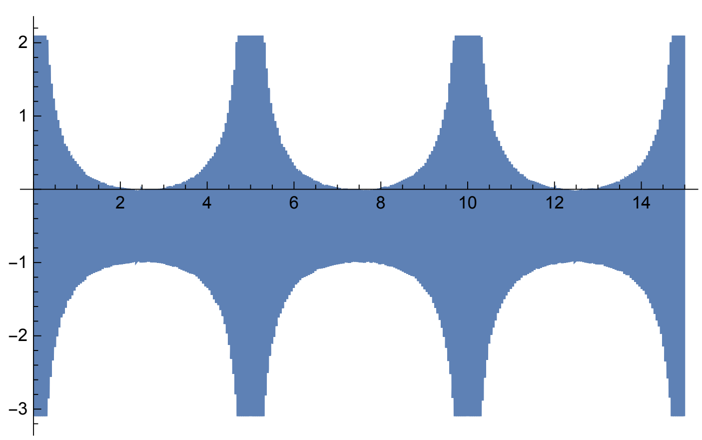
Out[39]=



The function could have all periods 5/n, and the corresponding values of p would be 2 π n/5. If we summed, say, the first 100 functions cos (2 n π x/5), we would get a function whose period is still 5 :

In[40]:= `Plot[Sum[Cos[2 π n x / 5], {n, 1, 100}], {x, 0, 15}]`

Out[40]=



And while a mess, you can see that the sum of functions still has a period of 5. The purpose of this exercise is to demonstrate that if we have a function that is 2L periodic, the sum of all sin and cos of the form sin(n π x/L) (or cos(n π x/L) produces a function whose period is equal to the original.

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5. Now we have a function that is 2 L periodic on [0, 1]. This means that 2 L = 1 and we will use L = 1/2.

We can see that the average value of the function is zero on the interval, so we expect that a_o is zero. For a_n and b_n we write:

$$a_n = \frac{1}{L}\int_0^1 f(x) \operatorname{Cos}[n \pi x / L]\, dx$$

With L = 1/2 :

$$= 2\int_0^{1/2} -\cos(2 n \pi x)\, dx + 2\int_{1/2}^{1} \cos(2 n \pi x)\, dx = 0$$

(each integral returns sin (2 n π x) evaluated at either 0, 1/2, or 1)

$$b_n = 2\int_0^{1/2} -\sin(2 n \pi x)\, dx + 2\int_{1/2}^{1} \sin(2 n \pi x)\, dx$$

$$= \frac{2}{2 n \pi}(\cos(n \pi) - 1) - \frac{2}{2 n \pi}(\cos(2 n \pi) - \cos(n \pi))$$

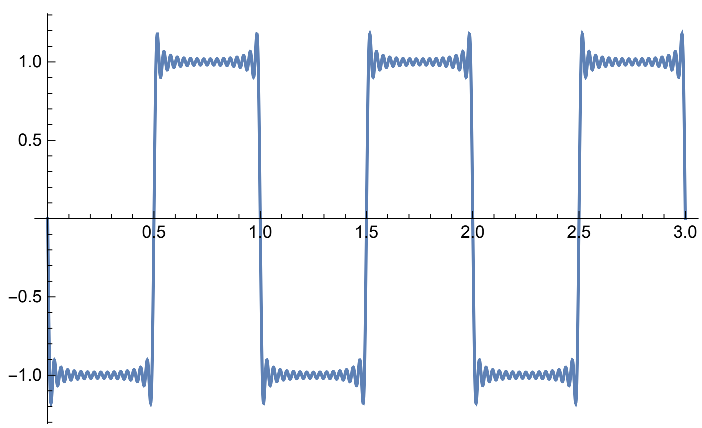
$$= \frac{1}{n \pi}[2(-1)^n - 2] = \frac{2}{n \pi}[(-1)^n - 1] = \begin{cases} 0, & n \text{ even} \\ -4/n\pi, & n \text{ odd} \end{cases}$$

Our Fourier series is :

$$f(x) = \Sigma b_n \sin(2 n \pi x) = \frac{-4}{\pi}\left[\operatorname{Sin}(2 \pi x) + \frac{\operatorname{Sin}(4 \pi x)}{2} + \frac{\operatorname{Sin}(6 \pi x)}{3} + \ldots\right]$$

Verifying :

In[60]:=
```
Plot[(-4 / π) Sum[Sin[2 n π x] / n, {n, 1, 31, 2}], {x, 0, 3}]
```

Out[60]=


and we verify over three cycles.

6. We consider the function :

$$f(x) = \begin{cases} 10, & 0 < x < 10 \\ 20, & 10 < x < 20 \end{cases}$$

The average value of the function on the interval is 15, so $a_o = 15$. The function is 2L periodic on [0,20], so L = 10. Computing the other coefficients:

In[67]:=
```
Clear[a, b]
a = (1/10) Integrate[10 Cos[n π x/10], {x, 0, 10}] +
  Integrate[20 Cos[n π x/10], {x, 10, 20}]
```

Out[68]= $\frac{10\,\mathrm{Sin}[n\pi]}{n\pi} + \frac{200\,(-\mathrm{Sin}[n\pi] + \mathrm{Sin}[2n\pi])}{n\pi}$

In[63]:= $\frac{10\,\mathrm{Sin}[n\pi]}{n\pi} + \frac{200\,\mathrm{Sin}[2n\pi]}{n\pi}$

and knowing the properties of sin, we can see these are all zero.

In[71]:=
```
b = (1/10)
  (Integrate[10 Sin[n π x/10], {x, 0, 10}] + Integrate[20 Sin[n π x/10], {x, 10, 20}])
```

Out[71]= $\frac{1}{10}\left(-\frac{100\,(-1+\mathrm{Cos}[n\pi])}{n\pi} + \frac{200\,(\mathrm{Cos}[n\pi] - \mathrm{Cos}[2n\pi])}{n\pi}\right)$

Let' s figure out what this means. The first term on the left in the output is zero for even n, and equal to 20/(n π) for odd n. The term on the right is also zero for even values of n, and equals -40/(n π) for odd values. So we can write :

$$b_n = \begin{cases} 0, & n\ \text{even} \\ \frac{-20}{n\pi}, & n\ \text{odd} \end{cases}$$

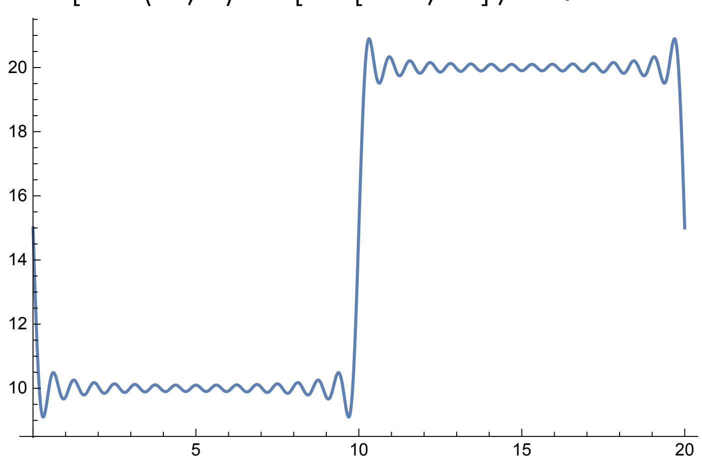
And our Fourier series is:

$$f(x) = 15 - \frac{20}{\pi}\left[\mathrm{Sin}(\pi x/10) + \frac{\mathrm{Sin}(3\pi x/10)}{3} + \frac{\mathrm{Sin}(5\pi x/10)}{5} + \ldots\right]$$

Verifying :

In[73]:=
```
Plot[15 - (20/π) Sum[Sin[n π x/10]/n, {n, 1, 31, 2}], {x, 0, 20}]
```

Out[73]=



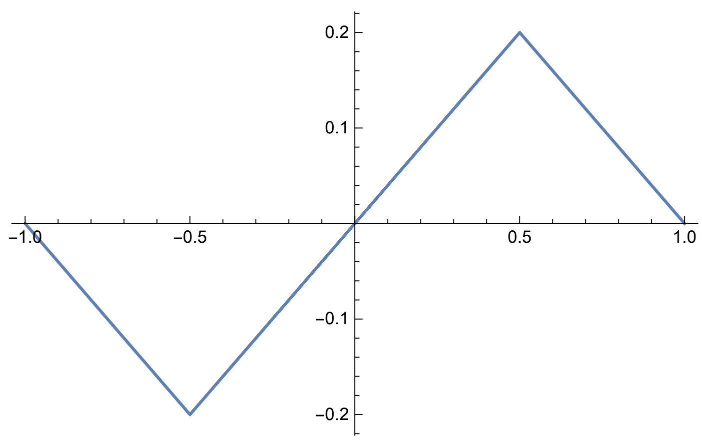
7. This is an example where we don' t have a periodically repeating form, and we have to extend the function from [0, L] to [-L, L] to compute its Fourier series. As we will see later in the course, the initial conditions demand a sine series, so our extension becomes :

$$fx) = \begin{cases} 2hx/L, & -L/2 < x < L/2 \\ 2h - 2hx/L, & L/2 < x < L \\ -2h - 2hx/L, & -L < x < -L/2 \end{cases}$$

Writing the piecewise function using the “Which” command and plotting, we see:

In[78]:=
```
Clear[f, h, L]
h = 0.2; L = 1;
f[x_] := Which[-L < x < -L/2, -2 h - 2 h x / L,
  -L/2 < x < L/2, 2 h x / L, L/2 < x < L, 2 h - 2 h x / L]
Plot[f[x], {x, -L, L}]
```

Out[81]=



And remember I have to input values for h and L to produce a plot. We have an odd function on [-L,L], so we know our Fourier series will have only odd (sin) terms. We need only compute the b_n coefficients:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin(n\pi x/L)\,dx$$

We only need to integrate from 0 to L, so we only need that portion of the extended function that corresponds to the actual string. Getting some help here:

In[96]:=
```
Clear[f, h, L]

Simplify[(2/L) (Integrate[2 h x / L Sin[n π x / L], {x, 0, L/2}] + Integrate[
    (2 h - 2 h x / L) Sin[n π x / L], {x, L/2, L}]), Assumptions → n ∈ Integers]
```

Out[97]=

$$\frac{8\,h\,\text{Sin}\left[\frac{n\pi}{2}\right]}{n^2\,\pi^2}$$

And that’s pleasingly simple. The coefficients are zero for even n, and alternate sign for odd n, so our Fourier series is:

$$f(x) = \frac{8h}{\pi^2}\,\text{Sin}[\pi x / L] - \frac{\text{Sin}[3 \pi x / L]}{9} + \frac{\text{Sin}[5 \pi x / L]}{25} - \ldots]$$

Verifying :

In[101]:=
```
Clear[f, x, L, h]
h = 0.2; L = 1;
Plot[(8 h / π^2) Sum[Sin[n π / 2] Sin[n π x / L] / n^2, {n, 1, 31}], {x, -L, L}]
```

Out[103]=