PHYS 301
HOMEWORK #6-- Solutions

1. Since j is the repeated index in the expression, we have that

$$\delta_{ij}\,\delta_{jk} \;=\; \delta_{i1}\,\delta_{1\,k} + \delta_{i2}\,\delta_{2\,k} + \delta_{i3}\,\delta_{3\,k}$$

The entire expression will equal zero is i ≠ k. But if i = k = 1, the first term is one and the other two terms are zero. Similarly, if i = k = 2 or 3, one term is equal to one and the other two are zero. Therefore, the expression is zero if i ≠ k and 1 if i = k, therefore,

$$\delta_{ij}\,\delta_{jk} \;=\; \delta_{ik}$$

We call this the contraction of two Kronecker deltas.

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2. Since i, j, k, l, and m are repeated indices, we are asked to to the quintiple sum of the product of permutation tensors. The first and last few terms of this sum are :

$$\epsilon_{ijklm}\,\epsilon_{ijklm} \;=\; \epsilon_{11\,111}\,\epsilon_{11\,111} \;+\; \epsilon_{11\,112}\,\epsilon_{11\,112} + \ldots\,\epsilon_{55\,554}\,\epsilon_{55\,554} \;+\; \epsilon_{55\,555}\,\epsilon_{55\,555}$$

Since there are a total of $5^5 = 3125$ terms, I will forego writing them all out. We know that the vast bulk of these terms will be zero and that the only non zero terms will be those where all the indices are different. Now, there are still quite a few of these (120 in fact) but we don't need to write them all out. We need only to figure out how many ways we can write these separate indices. We have five ways to choose the first index, four ways to choose the second index, three ways...and so on, so the total number of ways of writing five different numbers is 5·4·3·2·1 = 5! = 120. While some of these permutations are even and some are odd, we don't even need to figure out which is which; we are summing the product of two epsilons, so either we are multiplying 1·1 or (-1)(-1). Since we are adding 120 of these products, we have finally that:

$$\epsilon_{ijklm}\,\epsilon_{ijklm} \;=\; 120 \;=\; 5!$$

That' s why it' s called the permutation tensor.

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3. First, we write the identity in terms of summation notation :

$$\nabla \times (\mathbf{A} \times \mathbf{B}) \;=\; \epsilon_{mni}\,\frac{\partial}{\partial x_n}(\epsilon_{ijk}\,A_j\,B_k)$$

Since ϵ is a constant with respect to any spatial coordinate, it can be moved out of the differential operator giving us :

$$\epsilon_{mni}\,\epsilon_{ijk}\,\frac{\partial}{\partial x_n}(A_j\,B_k) \;=\; \epsilon_{imn}\,\epsilon_{ijk}\,\frac{\partial}{\partial x_n}(A_j\,B_k)$$

The last step follows from cyclically permuting the indices. This expression will provide the m^{th} component of the required vector.

Notice that we will be using the ϵ-δ relationship, and also that we have the derivative of a product. We obtain:

$$\epsilon_{imn}\,\epsilon_{ijk}\,\frac{\partial}{\partial x_n}(A_j\,B_k) =$$

$$(\delta_{mj}\,\delta_{nk} - \delta_{mk}\,\delta_{nj})\,\frac{\partial}{\partial x_n}(A_j\,B_k) = (\delta_{mj}\,\delta_{nk} - \delta_{mk}\,\delta_{nj})\left(A_j\,\frac{\partial}{\partial x_n}B_k + B_k\,\frac{\partial}{\partial x_n}A_j\right) =$$

$$\delta_{mj}\,\delta_{nk}\left(A_j\,\frac{\partial}{\partial x_n}B_k + B_k\,\frac{\partial}{\partial x_n}A_j\right) - \delta_{mk}\,\delta_{nj}\left(A_j\,\frac{\partial}{\partial x_n}B_k + B_k\,\frac{\partial}{\partial x_n}A_j\right)$$

the first term will be zero unless m = j and n = k. The second term will be zero unless m = k and n = j. Making these substitutions yields:

$$A_m\,\frac{\partial}{\partial x_n}B_n + B_n\,\frac{\partial}{\partial x_n}A_m - \left(A_n\,\frac{\partial}{\partial x_n}B_m + B_m\,\frac{\partial}{\partial x_n}A_n\right)$$

Now we' re ready for the last step, converting this back to vector notation. Notice the repeated index in each term, this will tell us where the appropriate dot product will occur. The expression above becomes:

$$A_m\,(\nabla\cdot\mathbf{B}) + (\mathbf{B}\cdot\nabla)\,A_m - (\mathbf{A}\cdot\nabla)\,B_m + B_m\,(\nabla\cdot\mathbf{A})$$

Summing over all m components gives the identity:

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,(\nabla\cdot\mathbf{B}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B} - \mathbf{B}\,(\nabla\cdot\mathbf{A})$$

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4. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) \rightarrow \dfrac{\partial}{\partial x_i}(\epsilon_{ijk}\,A_j\,B_k)$

The expression in parentheses produces the i^{th} component of the cross product, and this component is dotted with with del operator. Since ϵ is a constant with respect to the derivative, we can write:

$$\frac{\partial}{\partial x_i}(\epsilon_{ijk}\,A_j\,B_k) = \epsilon_{ijk}\,\frac{\partial}{\partial x_i}(A_j\,B_k)$$

Performing the indicated differentiation of a product:

$$\epsilon_{ijk}\,\frac{\partial}{\partial x_i}(A_j\,B_k) = A_j\left(\epsilon_{ijk}\,\frac{\partial}{\partial x_i}B_k\right) + B_k\left(\epsilon_{ijk}\,\frac{\partial}{\partial x_i}A_j\right)$$

Each term in parentheses represents a curl, but we need to be very careful about the order of indexing. The first term on the right produces the **i** × k^{th} component, which is the $(-j)^{\text{th}}$ component. The second term produces the **i** × j^{th} or +k component. Therefore, we can write:

$$\epsilon_{ijk}\,\frac{\partial}{\partial x_i}(A_j\,B_k) = A_j\left(\epsilon_{ijk}\,\frac{\partial}{\partial x_i}B_k\right) + B_k\left(\epsilon_{ijk}\,B_k\,\frac{\partial}{\partial x_i}A_j\right) = \mathbf{A}\cdot(-\nabla\times\mathbf{B}) + \mathbf{B}\cdot(\nabla\times\mathbf{A})$$

5. $\nabla\left(\frac{f}{g}\right) \rightarrow \frac{\partial}{\partial x_i}\left(\frac{f}{g}\right)$

Using the quotient rule of differentiation:

$$\frac{\partial}{\partial x_i}\left(\frac{f}{g}\right) = \frac{1}{g}\frac{\partial f}{\partial x_i} + f\left(\frac{-1}{g^2}\frac{\partial g}{\partial x_i}\right)$$

Summing over all components yields:

$$\frac{1}{g}\frac{\partial f}{\partial x_i} + f\left(\frac{-1}{g^2}\frac{\partial g}{\partial x_i}\right) = \frac{1}{g}(\nabla f) - \frac{f}{g^2}(\nabla f) = \frac{g(\nabla f) - f(\nabla g)}{g^2}$$

6. a) Writing the gradient :

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right)$$

And the curl is:

$$\nabla\times(\nabla\phi) = \left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}\right)\hat{x} - \left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial x}\right)\hat{y} + \left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x}\right)\hat{z}$$

Recall that for continuously differentiable functions, the order of differentiation is interchangeable, such as:

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

Each term in the curl is zero since it is the difference between two second partial derivatives of the form shown directly above. Therefore, the curl of any gradient is zero. Since a conservative force has a potential and can be written as the gradient of a scalar potential, this result is why $\nabla\times F = 0$ is a test for a conservative force.

b) Writing $\nabla\times(\nabla\phi)$ in summation notation :

$$\nabla\times(\nabla\phi) = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k}$$

Now, since the order of differentiation is interchangeable, we can write :

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k} = \epsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial\phi}{\partial x_j} \tag{1}$$

But we also know that from the properties of the permutation tensor that switching the order of indices changes the sign, so this means also that

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_k} = -\epsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial\phi}{\partial x_j} \tag{2}$$

Equations (1) and (2) imply that an expression equals its negative, this can occur only if the expression is zero, therefore we confirm that $\nabla\times(\nabla\phi) = 0$.