PHYS 301
HOMEWORK #7-- Solutions

1. We write the divergence of a vector v in spherical coordinates :

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\, v_r\right) + \frac{\partial}{\partial\theta}(r \sin\theta\, v_\theta) + \frac{\partial}{\partial\phi}(r\, v_\phi)\right]$$

We are interested in the position vector $\mathbf{r} = r\,\hat{\boldsymbol{r}}$. This vector has only an r component, so the divergence becomes:

$$\nabla \cdot \mathbf{r} = \frac{1}{r^2 \sin\theta}\,\frac{\partial}{\partial r}\left(r^2 \sin\theta \cdot r\right)$$

Since $\sin\theta$ is a constant with respect to $\partial/\partial r$, we can move it outside the derivative leaving us with :

$$\nabla \cdot \mathbf{r} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^3\right) = \frac{3\,r^2}{r^2} = 3$$

as we must obtain.

---

2. We begin with the Laplacian in spherical coordinates; since we only need the radial component, we can write :

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\,\frac{\partial V_r}{\partial r}\right)\right] = 0$$

For our trial solution of $V = c\, r^n$, we have:

$$\nabla^2 V = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} c\, r^n\right)\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\, n\, c\, r^{n-1}\right)\right] = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(n\, c\, r^{n+1}\right)$$

$$= \frac{1}{r^2} \cdot n\,(n+1)\,c\, r^n = 0$$

For this expression to equal zero, either r is always zero or the product $n(n+1) = 0$ which implies that n can be either 0 or -1.

For each of the next three problems, our trial solution will be:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

which implies the following derivatives:

$$y' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \quad \text{and } y'' = \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2}$$

To save myself some typing I will omit the upper limits of some of the summations in the solutions below.

---

3. $y'' - x y' + 2 y = 0$

We use our trial solutions:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n$$

Multiplying terms in the second summation yields :

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n$$

We re - index the first sum by setting k = n - 2 and obtain :

$$\sum_{n=0} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n$$

Finally, we strip out the n = 1 terms from the first and third sums to give :

$$2 a_2 + 2 a_o + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n + 2 a_n] x^n = 0$$

The individual terms tell us :

$$a_2 = - a_0$$

and the recursion relation becomes:

$$a_{n+2} = \frac{(n-2) a_n}{(n+2)(n+1)}$$

Notice that we expect both an odd and even branch of the solution. Notice also that the factor of n - 2 in the numerator means that our coefficient will be zero when n = 2, and therefore all higher order even coefficients will be zero. Using the recursion relation, we obtain for our coefficients :

$$a_2 = -a_o$$

$$a_4 = 0$$

$$a_3 = \frac{-1 a_1}{3 \cdot 2}$$

$$a_5 = \frac{a_3}{5 \cdot 4} = \frac{- a_1}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$a_7 = \frac{3 a_5}{7 \cdot 6} = \frac{- 3 a_1}{7!}$$

and we can write our solution as :

$$y = a_o \left(1 - x^2\right) + a_1 \left(x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{3 x^7}{7!} - ...\right)$$

---

4. $\left(x^2 + 4\right) y'' + x y = x + 2$

This is a case in which we have non-zero terms on the right hand side, so we will need to be careful to set all the x^0 terms on the left equal to 2, and all the x^1terms on the left to 1 (the coefficient of x on the right). Substituting our trial solutions gives us:

$$\left(x^2+4\right)\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} \;+\; x\sum_{n=0}^{\infty} a_n\,x^n \;=\; x+2$$

Performing the indicated multiplications:

$$x^2\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} \;+\; 4\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} \;+\; x\sum_{n=0}^{\infty} a_n\,x^n \;=\; x+2$$

or :

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n \;+\; 4\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} \;+\; \sum_{n=0}^{\infty} a_n\,x^{n+1} \;=\; x+2$$

Re - index by setting k = n - 2 in the second sum and k = n + 1 in the third sum yielding :

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n \;+4\sum_{n=0} (n+2)\,(n+1)\,a_{n+2}\,x^n \;+\; \sum_{n=1}^{\infty} a_{n-1}\,x^n \;=\; x+2$$

Now, to get all sums to the same lower limit of n = 2, we strip out the n = 0 and n = 1 terms from the middle sum, and the n = 1 term from the final sum :

$$8\,a_2+24\,a_3\,x \;+\; a_o\,x \;+\; \sum_{n=2}^{\infty}[4\,(n+2)\,(n+1)\,a_{n+2}+n\,(n-1)\,a_n+a_{n-1}]\,x^n \;=\; x+2$$

Now we equate all the terms on the left to their corresponding powers on the right. The summatin begins at n = 2, so we can write immediately that:

$$8\,a_2 \;=\; 2 \;\Rightarrow\; a_2 \;=\; \frac{1}{4}$$

Notice that we find an absolute value for a_2 without reference to any other coefficient. Also, we obtain:

$$24\,a_3+a_o \;=\; 1 \;\;\Rightarrow\; a_3 \;=\; \frac{1-a_o}{24} \;=\; \frac{1}{24}-\frac{a_0}{24}$$

Our recursion relation, valid for n ≥ 2 is:

$$a_{n+2} \;=\; -\frac{n\,(n-1)\,a_n}{4\,(n+2)\,(n+1)}-\frac{a_{n-1}}{4\,(n+2)\,(n+1)}$$

Let' s see what these recursion relations give :

$$a_4 \;=\; \frac{-1}{24}\,a_2-\frac{a_1}{48} \;=\; \frac{-1}{24}\left(\frac{1}{4}\right)-\frac{1}{48}\,a_1 \;=\; \frac{-1}{96}-\frac{1}{48}\,a_1$$

$$a_5 \;=\; \frac{-6}{80}\,a_3 \;-\; \frac{a_2}{80} \;=\; \frac{-3}{40}\left(\frac{1}{24}-\frac{a_o}{24}\right)-\frac{1}{4}\left(\frac{1}{80}\right) \;=\; \frac{-1}{160}+\frac{1}{320}\,a_o$$

Now let' s remember explicitly that our series solution is:

$$y \;=\; a_o \;+a_1\,x \;+\; a_2\,x^2 \;+\; a_3\,x^3 \;+a_4\,x^4 \;+a_5\,x^5+\ldots$$

$$y \;=\; a_o+a_1\,x+\frac{1}{4}\,x^2+\left(\frac{1}{24}-\frac{a_o}{24}\right)x^3 \;+\left(\frac{-1}{96}-\frac{1}{48}\,a_1\right)x^4+\left(\frac{-1}{160}+\frac{1}{320}\,a_o\right)x^5$$

I can group these according to coefficient and write:

$$y = a_o\left(1 - \frac{1}{24}x^3 + \frac{1}{320}x^5 - ...\right) + a_1\left(x - \frac{1}{48}x^3 + ...\right) + \left(\frac{1}{4}x^2 + \frac{1}{24}x^3 - \frac{1}{96}x^4 - \frac{1}{160}x^5 + ...\right)$$

Why do we have three parts to the solution? We expect a second order differential equation to have two branches, but what does the third branch represent?

---

5. $y'' + (x-1)\,y' + (2x-3)\,y = 0$

The trial solution yields :

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + (x-1)\sum_{n=1}^{\infty} n\,a_n x^{n-1} + (2x-3)\sum_{n=0}^{\infty} a_n x^n = 0$$

Multiplying gives :

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + \sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=1}^{\infty} n\,a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^{n+1} - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

We have to re - index the first, third and fourth summations, using, respectively, k = n - 2, k = n - 1 and k = n + 1. Making these subsitutions gives us :

$$\sum_{n=0} (n+2)(n+1)\,a_{n+2} x^n + \sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=0}^{\infty} (n+1)\,a_{n+1} x^n + 2\sum_{n=1}^{\infty} a_{n-1} x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

Stripping out the n = 1 terms from the sums with lower limits of zero (the first, third and last) results in :

$$2a_2 - a_1 - 3a_o = 0 \Rightarrow a_2 = \frac{a_1}{2} + \frac{3a_o}{2}$$

and our recursion relation becomes

$$a_{n+2} = \frac{a_{n+1}}{n+2} - \frac{(n-3)}{(n+2)(n+1)}a_n - \frac{2a_{n-1}}{(n+2)(n+1)}$$

Using the recursion relation, we find that:

$$a_3 = \frac{a_2}{3} - \frac{(-2)\,a_1}{3\cdot 2} - \frac{2a_o}{3\cdot 2} = \frac{a_2}{3} + \frac{a_1}{3} - \frac{a_o}{3} = \frac{1}{3}\left(\frac{a_1}{2} + \frac{3a_o}{2}\right) + \frac{a_1}{3} - \frac{a_o}{3} = \frac{a_1}{2} + \frac{1}{6}a_o$$

$$a_4 = \frac{a_3}{4} - \frac{(-1)\,a_2}{12} - \frac{2a_1}{12} = \frac{1}{4}\left(\frac{a_1}{2} + \frac{1}{6}a_o\right) + \frac{1}{12}\left(\frac{a_1}{2} + \frac{3a_o}{2}\right) - \frac{a_1}{6} = \frac{1}{6}a_o$$

and our series solution becomes:

$$y = a_o\left(1 + \frac{3}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + ...\right) + a_1\left(x + \frac{x^2}{2} + \frac{x^3}{2} + ...\right)$$