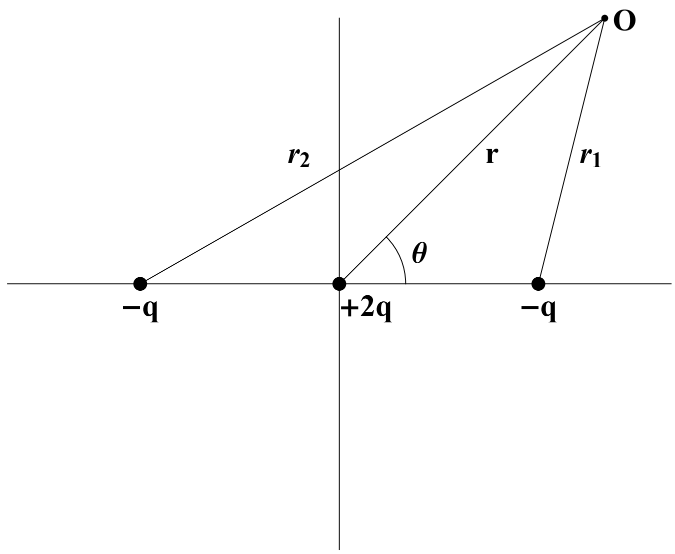
PHYS 301
HOMEWORK #9

Due : 12 April 2017

1. Consider the electric quadrupole in the diagram below :



The -q charges are located at (a,0) and (-a,0) along the x-axis. Express the potential at O due to this arrangement of charges in terms of Legendre polynomials. Assuming a<<r, what is the leading term of the expansion?

Solution : We know the total potential at O will be the sum of the three potentials :

$$V_T = V_1 + V_2 + V_3$$

Let' s call V_1, V_2, V_3 respectively the potentials due to the charge at a, -a, and the origin. Thus we can write:

$$V_1 = \frac{-kq}{r_1},\ V_2 = \frac{-kq}{r_2},\ V_3 = \frac{2kq}{r}$$

Using the law of cosines as shown in class, we can write $r_1$and r_2 as:

$$r_1 = \sqrt{r^2 + a^2 - 2\,a\,r\cos\theta} = r\sqrt{1 + (a/r)^2 - 2\,(a/r)\cos\theta}$$

$$r_2 = \sqrt{r^2 + a^2 - 2\,a\,r\cos(180 - \theta)} = r\sqrt{1 + (a/r)^2 + 2\,(a/r)\cos\theta}$$

Using these results for the denominators of V_1 and V_2:

$$V_1 = \frac{-k q}{r} \cdot \frac{1}{\sqrt{1 + (a/r)^2 - 2 (a/r) \cos\theta}} = \frac{-k q}{r} \sum_{m=0}^{\infty} P_m (\cos\theta) (a/r)^m$$

$$V_2 = \frac{-k q}{r} \cdot \frac{1}{\sqrt{1 + (a/r)^2 + 2 (a/r) \cos\theta}} = \frac{-k q}{r} \sum_{m=0}^{\infty} P_m (\cos\theta) (-a/r)^m$$

$$= \frac{-k q}{r} \sum_{m=0}^{\infty} (-1)^m P_m (\cos\theta) (a/r)^m$$

If we add the three potentials, we will obtain:

$$V_T = \frac{-k q}{r} \left[\sum_{m=0}^{\infty} P_m (\cos\Theta) (a/r)^m + \sum_{m=0}^{\infty} (-1)^m P_m (\cos\theta) (a/r)^m \right] + \frac{2 k q}{r}$$

Now consider the sum of Legendre series. In the second summation, $(-1)^m$ is even if m is even, and is odd if m is odd. When m is even, the two sums add; when m is odd, the two sums cancel. Therefore, we can write the total potential as:

$$V_T = \left[\frac{-2 k q}{r} \sum_{\text{even}}^{\infty} P_m (\cos\theta) (a/r)^m \right] + \frac{2 kq}{r}$$

Let' s write out the first few terms of the summation :

$$V_T = -\frac{2 kq}{r} \left[P_o (\cos\theta) (a/r)^0 + P_2 (\cos\theta) (a/r)^2 + ... \right] + \frac{2 k q}{r}$$

But since P_0 =1, the first term in the summation is 1 and cancels the 2 k q/r term, so the lead term in the expansion of the quadrupole is:

$$V_T = \frac{-2 k q}{r} \cdot \frac{1}{2} \left(3 \cos^2\theta - 1\right) (a/r)^2 + .. = \frac{-k q a^2}{r^3} \left(3 \cos^2\theta - 1\right)$$

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2. Determine the Legendre coefficients out to c_5 for the function:

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

and write out the Legendre series for this function.

Solution : First we notice that the function is an odd function. This allows us to make use of symme try. Since Legendre polynomials are even/odd if they are of even/odd order, we know that for an odd function f :

$$\int_{-1}^{1} f(x) P_m (x)\, dx = 0 \text{ if m is even and } 2 \int_{0}^{1} f(x) P_m (x)\, dx \text{ if m is odd.}$$

Therefore, this function will have only odd coefficients, which we compute via :

$$c_m = \frac{2 m + 1}{2} \int_{-1}^{1} f(x) P_m (x)\, dx$$

For our odd function, we have:

$$c_1 = \frac{3}{2}\cdot 2\int_0^1 1\cdot x\, dx = \frac{3}{2}$$

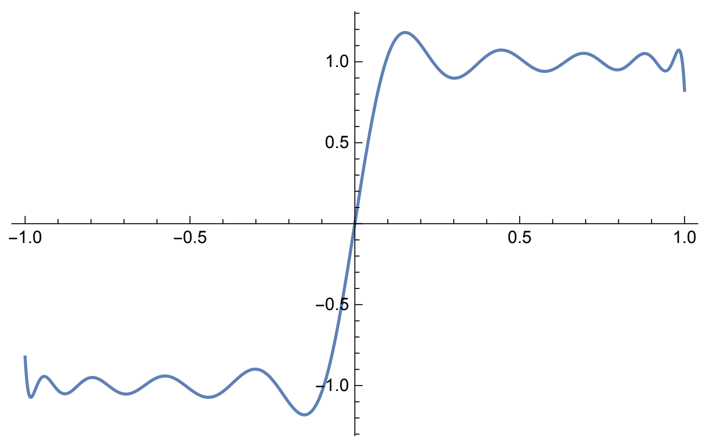
$$c_3 = \frac{7}{2}\cdot 2\int_0^1 1\cdot \frac{1}{2}\left(5x^3 - 3x\right) dx = \frac{-7}{8}$$

$$c_5 = \frac{11}{2}\cdot 2\int_0^1 1\cdot \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) dx = \frac{11}{8}\left[\frac{63}{6} - \frac{70}{4} + \frac{15}{2}\right] = \frac{11}{16}$$

giving us our Legendre series:

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x)$$

A plot of the Legendre series out to m = 19 shows that the series does converge slowly to the function:



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3. Expand the following as Legendre series; you may use Mathematica to verify your results, but you must show all integration by hand. (10 pts for each series.

a) $x^2 - x$

Solution : Even though the function is neither even nor odd, we can still make use of symmetry by realizing only the even part will contribute to even order coefficients, and only the odd part will contribute to odd order coefficients. So we can write :

$$c_0 = \frac{1}{2}\cdot 2\int_0^1 x^2\, dx = \frac{1}{3} \qquad c_1 = \frac{3}{2}\cdot 2\int_0^1 -x\cdot x\, dx = -1 \qquad c_2 = \frac{5}{2}\cdot 2\int_0^1 \frac{1}{2}\left(3x^2 - 1\right)x^2\, dx = \frac{2}{3}$$

And our Legendre series is:

$$f(x) = \frac{1}{3}P_0(x) - P_1(x) + \frac{2}{3}P_2(x)$$

We didn' t even need to use calculus, we could have just written:

$$x^2 - x = a_o P_o + a_1 P_1 + a_2 P_2 = 1 a_o + a_1 x + \frac{1}{2} a_2 (3x^2 - 1)$$

We know that coefficients on the left equal coefficients on the right, so that:

$$\frac{1}{2} a_2 (3x^2) = 1 \Rightarrow a_2 = \frac{2}{3}$$

$$a_1 x = -x \Rightarrow a_1 = -1$$

$$a_o - \frac{1}{2} a_2 = 0 \Rightarrow a_o = \frac{1}{3}$$

and we verify the coefficients obtained from the complete Legendre series calculation.

b) $7x^4 - 3x + 1$

Solution : Again making use of symmetry :

$$c_o = \frac{1}{2} \cdot 2 \int_0^1 7x^4 + 1 \, dx = \frac{7}{5} + 1 = \frac{12}{5}$$

$$c_1 = \frac{3}{2} \cdot 2 \int_0^1 -3x \cdot x \, dx = -3$$

$$c_2 = \frac{5}{2} \cdot 2 \int_0^1 (7x^4 + 1) \cdot \frac{1}{2} (3x^2 - 1) \, dx = \frac{5}{2} \left[\frac{21}{7} - \frac{7}{5} + 1 - 1 \right] = 4$$

$$c_3 = \frac{7}{2} \cdot 2 \int_0^1 (-3x) \cdot \frac{1}{2} \cdot (5x^3 - 3x) \, dx = \frac{-21}{2} [1 - 1] = 0$$

(we should have been able to have predicted this result; since the odd part of the function only goes to x, we should not expect any odd Legendre polynomials of order greater than 1).

$$c_4 = \frac{9}{2} \cdot 2 \int_0^1 (7x^4 + 1) \cdot \frac{1}{8} \cdot (35x^4 - 30x^2 + 3) \, dx = \frac{9}{8} \left[\frac{7 \cdot 35}{9} - \frac{210}{7} + \frac{21}{5} + 7 - 10 + 3 \right]$$

$$= \frac{9}{8} \left[\frac{245}{9} - 30 + \frac{21}{5} \right] = \frac{9}{8} \left[\frac{1414}{45} - 30 \right] = \frac{9}{8} \cdot \frac{64}{45} = \frac{8}{5}$$

and the Legendre series is:

$$f(x) = \frac{12}{5} P_o - 3 P_1 + 4 P_2 + \frac{8}{5} P_4$$

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4. The generating function for Bessel' s functions of the first kind is :

$$g(x, t) = e^{(x/2)(t - 1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

where J_n is the nth order Bessel function. Use the generating function to show that:

a) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

Solution: We start by differentiating the generating function. Since our identity does not involve derivatives of J, let’s start by taking the partial with respect to t:

$$\frac{\partial g}{\partial t} = \left(\frac{x}{2}\right)\left(1+\frac{1}{t^2}\right) e^{(x/2)(t-1/t)} = \frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Notice that the exponential term is just the generating function, so we can write:

$$\left(\frac{x}{2}\right)\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Distributing the terms on the left and performing the differentiation on the right:

$$(x/2)\left[\sum_{n=-\infty}^{\infty} J_n(x)\, t^n + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-2}\right] = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

Multiplying through by 2/x:

$$\left[\sum_{n=-\infty}^{\infty} J_n(x)\, t^n + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-2}\right] = \sum_{n=-\infty}^{\infty} \frac{2\,n}{x} J_n(x)\, t^{n-1}$$

Now we need to compare terms with the same exponent. Let’s choose the t^3 term and ask what value of n we need to choose in each summation to produce the t^3 term. In the first sum on the left, we set n = 3; in the second sum on the left n = 5; in the sum on the right, n =4. This gives us in the specific case:

$$J_3(x) + J_5(x) = \frac{2\,(4)}{x} J_4(x)$$

With no loss of generality I can set 4 = n, and the recursion relation becomes:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2\,n}{x} J_n(x)$$

b) $J_{n-1}(x) - J_{n+1}(x) = 2\,\frac{dJ_n(x)}{dx}$

Solution : Here we take the partial derivative with respect to x :

$$\frac{\partial g}{\partial x} = \frac{1}{2}\left(t-\frac{1}{t}\right) e^{(x/2)(t-1/t)} = \frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

The exponential term is, again, the generating function, so that:

$$\frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

where $J_n'(x) = \frac{dJ_n(x)}{dx}$

Multiply each side by 2, and distribute the t terms:

$$\sum_{n=-\infty}^{\infty} J_n(x)\, t^{n+1} - \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-1} = 2\sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

Equating like powers of t yields the recursion:

$$J_{n-1}(x) - J_{n+1}(x) = 2\,\frac{dJ_n(x)}{dx}$$

10 pts for each part.