Vector Analysis Using Mathematica

Overview

We have already seen how to use Mathematica for several different types of vector operations. We know that the dot and cross products of two vectors can be found easily as shown in the following examples :

```
In[28]:= Dot[{1, 2, 3}, {-1, 6, 4}]

Out[28]= 23
```

```
In[29]:= Cross[{1, 2, 3}, {-1, 6, 4}]

Out[29]= {-10, -7, 8}
```

Be sure to recognize the characteristics of Mathematica formatting; all functions start with a capital letter, function calls use square brackets "[ ]", and lists of numbers, in this case the components of the vectors
(1, 2, 3) and (-1, 6, 4) are in braces (the curly brackets "{ }"). A comma separates the two lists. Finally, remember that you have to engage SHIFT ENTER simultaneously to produce the output.

A Simple Mathematica Program

We can write a rudimentary program to help us compute the triple product of three vectors. Let' s say our three vectors are $v_1 = (1, 2, 3)$; $v_2 = (-1, 6, 4)$; $v_3 = (2, -3, 5)$. We can write a simple program to determine the volume of the parallelepiped composed of these vectors:

```
In[30]:= Clear[v1,v2,v3]
         v1={1,2,3};
         v2={-1,6,4};
         v3={2,-3,5};
         Dot[v1,Cross[v2,v3]]

Out[34]= 41
```

The first line of the code above clears out any previous values that might have been calculated for our vectors, v1, v2, and v3. While we clearly realize we have not defined these vectors before, it is always good programming practice to clear a variable name before using it.

The next three statements define our three vectors using standard Mathematica code. The semi - colon at the end of each statement suppresses output; try writing and executing this program without using the semi - colons.

Since we have defined our vectors as variables, we no longer use braces (curly brackets) since we are not involving a list of components in our calculations.

Finally, we can show that this program gives us exactly the same result we would get if we performed the two vector operations sequentially :

In[35]:=
```
Cross[v1, v2]
```
Out[35]=
```
{-10, -7, 8}
```
In[36]:=
```
Dot[v3, {-10, -7, 8}]
```
Out[36]=
```
41
```
Let me use this opportunity to introduce you to another Mathematica operation; the use of the "%" symbol. Using "%" allows you to take the last output generated and use that in a new calculation. So, if we reproduce the calculation from above, we get the cross product :

In[37]:=
```
Cross[v1, v2]
```
Out[37]=
```
{-10, -7, 8}
```
Now, if we want to take the dot product of v3 and the vector we just computed, we could do as we did immediately above and type in the components of the vector, or we can make use of the "%" symbol as :

In[38]:=
```
Dot[v3, %]
```
Out[38]=
```
41
```
Mathematica will automatically substitute the most recent output for "%".

Loading Packages in Mathematica

Until now, all the functions we have called upon {Dot, Cross, Sin, Cos, Exp, Int and so forth} have all been immediately accessible from Mathematica. There are some functions and packages that are not used (by all users) so frequently, and so must be called individually and loaded into your current session. We do this using the "Needs" function, and to load the vector analysis package, we proceed as :

In[39]:=
```
Needs["VectorAnalysis`"]
```
This will load the Vector Analysis package. (When you type this into the "Needs" call, there is no space between "Vector and "Analysis"; the "A" in "Analysis" is also capitalized). Notice that you make the function call with "Needs", and that the argument of the call is in quotes " ". Note also that the "Needs" call requires the use of the "`" character (it' s on the key just below the ESC key in the upper left of your keyboard).

Once you have typed in the complete "Needs" call, you hit SHIFT ENTER simultaneously to load the package. The last step is critical and is often overlooked. Only when you hit SHIFT ENTER do you actually load the package for your session.

Grad, Div, Curl in Mathematica

There are a few ways to write vector operations in Mathematica; I will proceed with the one I find easier and more understandable to use.

The Gradient

Let' s begin by finding the gradient of the function $x^2\, y^3\, z^4$.

In[49]:=
```
Grad[x^2 y^3 z^4, Cartesian[x, y, z]]
```
Out[49]= $\{2 x y^3 z^4, 3 x^2 y^2 z^4, 4 x^2 y^3 z^3\}$

When inputting expressions like the one above, make sure you remember to leave a space to indicate multiplication. Mathematica will automatically assume that spaces between variables indicate multiplication. So the expression : x y z means "x times y times z"; but the expression : xyz (no spaces) means the variable named "xyz".

Since Mathematica can compute vector properties in any coordinate system, it is necessary to indicate the system you are using.

The output gives you the components of the gradient vector. In standard format, we would write this gradient as:

$$\nabla f = 2 xy^3 z^4 \hat{x} + 3 x^2 y^2 z^4 \hat{y} + 4 x^2 y^3 z^4 \hat{z} \tag{1}$$

We can also define vectors as variables; as in :

In[50]:=
```
Clear[f]
f = x^2 y^3 z^4;
Grad[f, Cartesian[x, y, z]]
```
Out[52]= $\{2 x y^3 z^4, 3 x^2 y^2 z^4, 4 x^2 y^3 z^3\}$

Again, it is always wise to clear the variable name, and remember to leave spaces to represent multiplication.

Divergence

The gradient of a function results then the del operator acts on a scalar producing a vector gradient. The divergence of a function is the dot product of the del operator and a vector valued function producing a scalar. When we use Mathematica to compute Div, we must remember to input the components of a vector. If we wish to find the divergence of the gradient found above, we would write:

```
Div[{2 x y^3 z^4, 3 x^2 y^2 z^4, 4 x^2 y^3 z^3}, Cartesian[x, y, z]]
```
Out[53]= $12 x^2 y^3 z^2 + 6 x^2 y z^4 + 2 y^3 z^4$

And obtain the scalar $12 x^2 y^3 z^2 + 6 x^2 y z^4 + 2 y^3 z^4$.

We can combine operations and take the divergence of the gradient of f as defined above. In this case we do not clear f.

In[57]:=
```
Div[Grad[f, Cartesian[x, y, z]], Cartesian[x, y, z]]
```
Out[57]= $12 x^2 y^3 z^2 + 6 x^2 y z^4 + 2 y^3 z^4$

Notice that we must use Cartesian[x, y, z] in both the calculation of tne Grad and the Div.

Curl

The Curl of a vector results from taking the cross product of the del operator with a vector. The curl operation produces another vector. Let' s examine this by taking the curl of the vector :

$$\mathbf{v} = xyz\,\hat{\mathbf{x}} + xyz\,\hat{\mathbf{y}} + xyz\,\hat{\mathbf{z}}$$

We would write :

```
In[59]:= Curl[{x y z, x y z, x y z}, Cartesian[x, y, z]]
Out[59]= {-x y + x z, x y - y z, -x z + y z}
```

The output tells us that the curl is the vector :

$$\nabla \times \mathbf{v} = (xz - xy)\,\hat{\mathbf{x}} + (xy - yz)\,\hat{\mathbf{y}} + (yz - xz)\,\hat{\mathbf{z}}$$

Second Derivatives

Let' s use what we have learned to examine some other vector identities. In class, we proved that $\nabla\cdot(\nabla\times\mathbf{v})=0$ for all vectors. Let's see if results from *Mathematica* are consistent with this. Let's establish a vector $f = x^n y^n z^n \hat{x} + x^n y^n z^n \hat{y} + x^n y^n z^n \hat{z}$:

```
In[67]:= Clear[f]
In[68]:= f = {x^n y^n z^n, x^n y^n z^n, x^n y^n z^n};
```

(When inputting these components, remember to leave spaces between each variable name). First, let' s take the curl of this vector :

```
In[69]:= Curl[f, Cartesian[x, y, z]]
```

Out[69]= $\{-n\,x^n y^n z^{-1+n} + n\,x^n y^{-1+n} z^n,\ n\,x^n y^n z^{-1+n} - n\,x^{-1+n} y^n z^n,\ -n\,x^n y^{-1+n} z^n + n\,x^{-1+n} y^n z^n\}$

Now, if we take the div of this curl :

```
In[70]:= Div[Curl[f, Cartesian[x, y, z]], Cartesian[x, y, z]]
Out[70]= 0
```

As we expect.

Grad, Div, Curl in Other Coordinate Systems

We will spend much more time on this in a few weeks when we study different coordiante systems, but it is introduce some of this material now. If r is the position vector, we could take the gradient of r^2 in Cartesian coordinates by writing:

$$r^2 = x^2 + y^2$$

and so :

```
In[72]:= ∇(r^2) = Grad[x^2 + y^2, Cartesian[x, y, z]]
Out[72]= {2 x, 2 y, 0}
```

And get the vector $2x\,\hat{x} + 2y\,\hat{y}$. We could also find the gradient for this function in cylindrical coordinates:

```
In[73]:= Grad[r^2, Cylindrical[r, theta, z]]
Out[73]= {2 r, 0, 0}
```

Which is identical to the vector {2 x, 2 y, 0}.

We can find the gradient in spherical polar coordinates :

In[74]:=
```
Grad[r^2, Spherical[r, theta, phi]]
```

Out[74]= {2 r, 0, 0}

Plotting Vector Fields

In order to plot vector fields in Mathematica, we have to load another package :

In[75]:=
```
Needs["VectorFieldPlots`"]
```

Again, the format requires the "Needs" function; calling the particular package (in this case VectorField Plots), and using quotes and the "`" mark as needed. Remember also to hit SHIFT ENTER to load the package.

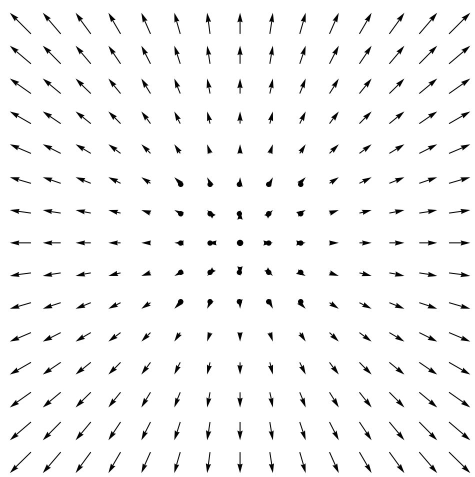
If you search for Vector Field Plots in the online documentation, you will get a list of the types of plots you can obtain. Let' s plot the field for the vector :

$$\mathbf{v} = \mathbf{x}\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

We call :

In[76]:=
```
VectorFieldPlot[{x, y}, {x, -3, 3}, {y, -3, 3}]
```

Out[76]=

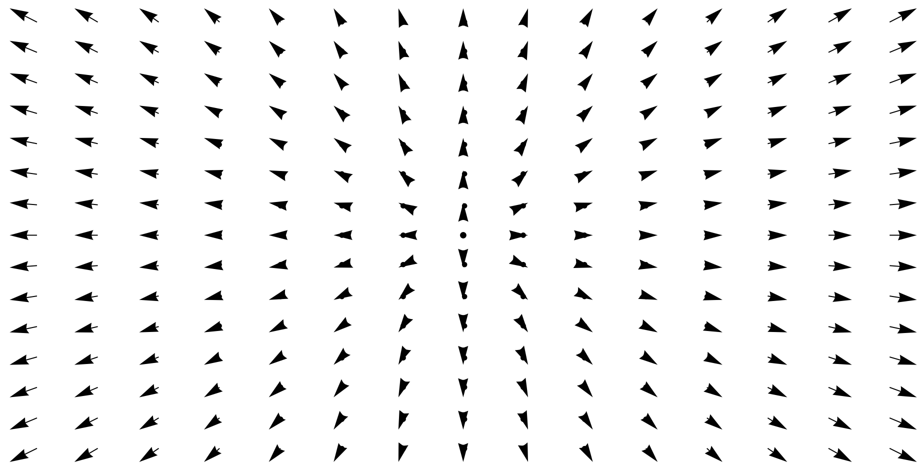


Be sure you understand all the formatting in the plot statement above. The function called is "VectorFieldPlot". The elements in the square bracket are the components of the vector, {x, y}, and the plot ranges. {x, -3, 3} instructs the program to plot this vector field from - 3 to 3 in the x direction, and {y, -3, 3} instructs similar in the y direction. Note that all lists are within braces.

There is no requirement that the plot ranges are the same in both dimensions, as shown in :

In[79]:=
```
VectorFieldPlot[{x, y}, {x, -2, 2}, {y, -1, 1}]
```

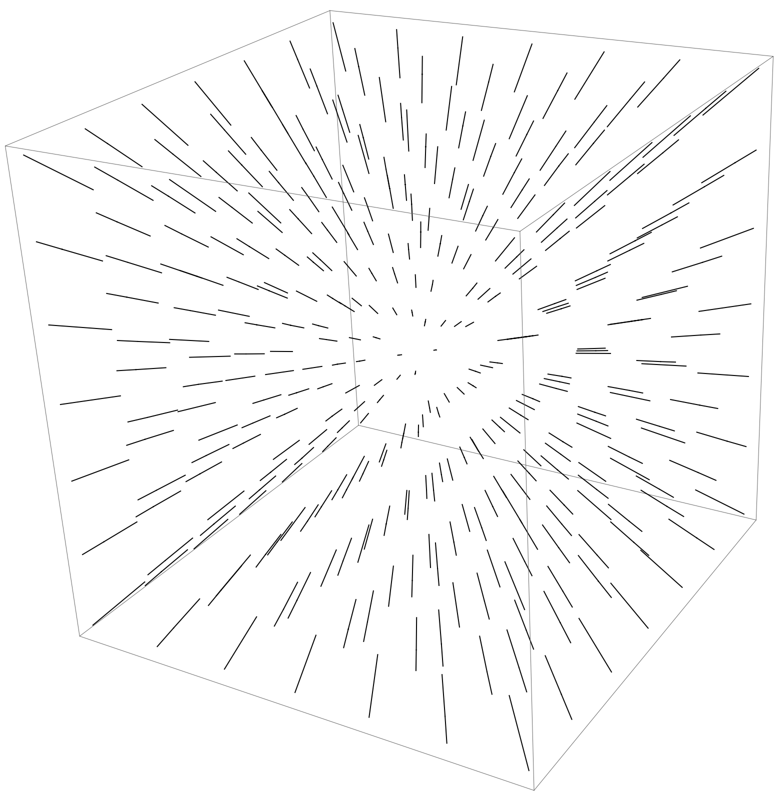
Out[79]=



A slightly different call allows us to plot vectors in 3 dimensions, as in this field radiating from the origin :

In[80]:=
```
VectorFieldPlot3D[{x, y, z}, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]
```

Out[80]=



The changes here should be self - evident; adding the "3D" descriptor and adding the plot range in z.

You should create this or other 3 D vector plots in Mathematica. If you put the cursor over the diagram, you will see the cursor indicate rotation; left click on the vector plot, and see that you can rotate the vector field to view it from all directions.

I strongly recommend searching for : VectorFieldPlots/guide/VectorFieldPlottingPackage in the online documentation center, and trying out the various types of plots (including gradient plot, gradient 3D plot, and so on).