PHYS 314
FIRST HOUR EXAM

Spring 2017

This is a closed book, closed note exam. You will not need nor be allowed to use calculators or other electronic devices on this test. At this time, store all electronic devices, including cell phones, out of sight.

Do all your writing in your blue book (s) making sure you put your name on each blue book you use. You may do questions in any order; please make sure to label clearly the question you are solving. Your answers must show clear and complete solutions; no work = no credit. The numbers in parentheses indicate the value of the question.

There is a list of equations and results at the end of the test. If you want to know any equation that does not appear on this list, ask and I will tell you (unless it's the answer to a question).

Note that there are options for extra credit below; 100% will be the maximum possible score.

1. An object is dropped from rest above the surface of the Earth. Assume atmospheric friction creates a resisting force of magnitude k **v** where k is a positive constant.

a) Write Newton' s second Law for this situation. (5)

Solution : Let' s choose down as the positive direction and we have :

$$m \frac{dv}{dt} = m g - k v$$

b) Solve the appropriate differential equation and obtain an expression for v (t). (15)

Solution : We solve by separation of variables :

$$\frac{m\,dv}{m g - k v} = dt \Rightarrow m \int \frac{dv}{m g - k v} = \int dt$$

$$\frac{-m}{k} \ln\left| m g - k v \right| = t + C \Rightarrow \ln |m g - k v| = \frac{-k t}{m} + C$$

(A constant times other constants is still a constant). Exponentiate both sides :

$$m g - k v = A e^{-k t/m}$$

where A is a constant. We evaluate A by using the initial condition that v = 0 when t = 0; using these values we obtain:

$$m g - 0 = A \Rightarrow A = m g$$

and our complete solution becomes:

$$m g - k v = m g e^{-kt/m} \Rightarrow v = \frac{m g}{k}\left(1 - e^{-k t/m}\right)$$

c) Use initial conditions to solve for any constants of integration, and use them to write the complete equation for v (t) (10)

d) *Extra credit* : If k is small, show that this expression approaches the value you would expect in the no friction case, namely v(t) = g t. (10)

Solution : In the limit that k -> 0, we expand the exponential and get :

$$v(t) = \frac{m g}{k}\left(1 - \left(1 - \frac{k t}{m}\right)\right) = \frac{m g}{k}\left(\frac{k t}{m}\right) = gt$$

as expected.

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2. At t = 0, an open top freight railroad car is moving along level, straight, frictionless tracks at a speed of v_o. The mass of the car before any rain collects in it is is M_o. At t = 0, the car enters a region of rain; since the car is moving into a region of increasing precipitation, the rate at which rain fills the car is given by $\alpha\, t^2$ where α is a positive constant with units kg / s^2. Assume the rain falls vertically with respec to the Earth.

a) Can you use the conservation of linear momentum to analyze the motion of this car? Can you use the conservation of energy? Explain both answers. (5)

Solution : Since there are no external forces acting on the rain/car system, we can use conservation of linear momentum. We cannot use conservation of energy since the collisions between the rain and the car are inelastic.

b) Write Newton' s second law for the car as it collects rain. (15)

Solution : This is an example of a variable mass problem, since the railroad car' s mass increases over time. Since there are no external forces acting on the system, the RHS of this equation will be zero. We begin by writing :

$$\Sigma F = \frac{dp}{dt} = \frac{d(m\,v)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = 0$$

We are told that $m(t) = M_o + \alpha\, t^2$ so $dm/dt = 2\,\alpha\,t$. At this point, we have two ways of solving this problem; one which explicitly uses conservation of momentum and one which does not. Let' s start with the method using conservation of momentum. Since momentum is conserved, we know that the product of m(t) v(t) is a constant, therefore we can write:

$$m\,v = M_o\,v_o \Rightarrow m = \frac{M_o\,v_o}{v}$$

Substituting this into our differential equation yields:

$$\frac{M_o\,v_o}{v}\frac{dv}{dt} + 2\,\alpha\,t\,v = 0 \Rightarrow \frac{1}{v^2}dv = -\frac{2\,\alpha\,t\,dt}{M_o\,v_o}$$

Integrating both sides:

$$\frac{-1}{v} = -\frac{\alpha\,t^2}{M_o\,v_o} + C \Rightarrow \frac{1}{v} = \frac{\alpha\,t^2}{M_o\,v_o} + C$$

Using the initial condition that $v = v_o$ when $t = 0$ yields:

$$\frac{1}{v_o} = C$$

Using this expression for the constant of integration:

$$\frac{1}{v} = \frac{\alpha\,t^2}{M_o\,v_o} + \frac{1}{v_o} = \frac{\alpha\,t^2 + M_o}{M_o\,v_o} \Rightarrow v = \frac{M_o\,v_o}{M_o + \alpha\,t^2}$$

We can also solve this problem without making explicit use of momentum conservation. Here, we explicitly use the expression for the time dependent mass of the car/rain system:

$$m(t) = M_o + \alpha\,t^2$$

in Newton' s second law, giving us :

$$\left(M_o + \alpha\,t^2\right)\frac{dv}{dt} + 2\,\alpha\,t\,v = 0$$

Separating variables gives us:

$$\frac{dv}{v} = -\frac{2\,\alpha\,t}{M_o + \alpha\,t^2}$$

Integrating both sides gives:

$$\ln v = -\ln\left|M_o + \alpha\,t^2\right| + C$$

Setting v = v_o at t = 0 tells us that C = ln v_o + ln M_o = ln (M_o v_o) using the properties of logs. Our solution becomes:

$$\ln v = -\ln\left|M_0 + \alpha\,t^2\right| + \ln(M_o\,v_o) = \ln\left[\frac{M_o\,v_o}{M_o + \alpha\,t^2}\right]$$

Exponentiating both sides gives us the same answer as before:

$$v = \frac{M_o\,v_o}{M_o + \alpha\,t^2}$$

c) Solve this differential equation (including constants of integration) to find an expression for v (t). (15)

d) *Extra credit* : Use the expression for v (t) to find an expression for distance traveled as a function of time. What is the limiting distance as t → ∞? (10)

Solution : To find the distance traveled as a function of time, we write v = dx/dt , separate variables and integrate :

$$v = \frac{dx}{dt} = \frac{M_o\,v_o}{M_o + \alpha\,t^2} \Rightarrow \int dx = \int v\,dt = \int \frac{M_o\,v_o}{M_o + \alpha\,t^2}\,dt$$

Using the integral provided at the end of the test, we know this becomes:

$$x = \frac{M_o\,v_o}{\sqrt{M_o\,\alpha}}\tan^{-1}\left(\sqrt{\frac{\alpha}{M_o}}\,t\right) + C$$

If we say the railroad car was at x = 0 when t = 0, the value of C = 0 and we have:

$$x(t) = v_o\sqrt{\frac{M_o}{\alpha}}\tan^{-1}\left(\sqrt{\frac{\alpha}{M_o}}\,t\right)$$

As t → ∞, the arc tangent -> $\pi/2$, so as the car becomes infinitely massive, its limiting distance traveled is:

$$x(t \to \infty) = \frac{\pi}{2}\,v_o\sqrt{\frac{M_o}{\alpha}}$$

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3. Another freight car, another railroad, another storm. An open top freight car, also of tare mass M_o, is moving along level, straight tracks with initial speed v_o at t = 0. At this time, rain begins to fall and collects at the rate of β t where β is a positive constant with units kg/s. This time, there is

friction with the tracks that generates a resisting force of magnitude γ **v** where γ is another positive constant. The rain falls vertically with respect to the Earth.

a) Can you use either the conservation of linear momentum or conservation of energy to analyze the motion of the car in this scenario? Explain both answers. (5)

Solution : Here, we can use neither conservation law since the collisions are inelastic and there is an external force (friction) acting on the system.

b) Write Newton' s second law for this situation. (10)

Proceding as we did above, except now there is an external force of friction:

$$(\mathrm{M_o} + \beta\, \mathrm{t})\, \frac{\mathrm{dv}}{\mathrm{dt}} + \beta\, \mathrm{v} = -\gamma\, \mathrm{v}$$

c) *Extra credit* : Solve this differential equation, use initial conditions to determine constant (s) of integration and write the expression for v (t). (10)

Solution : We can solve this via separation of variables :

$$\frac{\mathrm{dv}}{\mathrm{v}} = -\frac{(\beta+\gamma)\, \mathrm{dt}}{\mathrm{M_o} + \beta\, \mathrm{t}}$$

Integrating both sides:

$$\ln \mathrm{v} = -\frac{(\beta+\gamma)}{\beta} \ln \left| \mathrm{M_o} + \beta \mathrm{t} \right| + \mathrm{C}$$

$$\mathrm{v} = v_o \text{ at } \mathrm{t} = 0 \text{ imply that } \mathrm{C} = \ln \mathrm{v} + (\beta+\gamma) \ln (\mathrm{M_o})$$

or :

$$\mathrm{C} = \ln \mathrm{v} + \ln \mathrm{M_o}^{(\beta+\gamma)/\beta} = \ln\left[\mathrm{v}\, \mathrm{M_o}^{(\beta+\gamma)/\beta}\right]$$

and the expression for velocity becomes:

$$\ln \mathrm{v} = -(\beta+\gamma)/\beta \ln \left| \mathrm{M_o} + \beta\, \mathrm{t} \right| + \ln\left[\mathrm{v}\, \mathrm{M_o}^{(\beta+\gamma)/\beta}\right]$$

Adding the logs :

$$\ln \mathrm{v} = \ln\left[\frac{\mathrm{v}\, \mathrm{M_o}^{(\beta+\gamma)/\beta}}{(\mathrm{M_o} + \beta \mathrm{t})^{(\beta+\gamma)/\beta}}\right]$$

and upon exponentiating both sides:

$$\mathrm{v} = \frac{\mathrm{v}\, \mathrm{M}_0^{(\beta+\gamma)/\beta}}{(\mathrm{M_o} + \beta\, \mathrm{t})^{(\beta+\gamma)/\beta}}$$

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4. A damped harmonic oscillator obeys the following differential equation :

$$4\, \ddot{\mathrm{x}} + 32\, \dot{\mathrm{x}} + 100\, \mathrm{x} = 0$$

a) Write this ODE in the form :

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

and determine the values of β and ω_o for this system. (10)

Solution : Divide through by the leading term 4 :

$$\ddot{x} + 8\dot{x} + 25x = 0$$

This tells us that $2\beta = 8 \Rightarrow \beta = 4$ and $\omega_o^2 = 25$. Since $\omega_o^2 > \beta^2$ (restoring force dominates over friction) we expect to get oscillatory behavior; the system is underdamped.

b) Is the system underdamped, critically damped, or overdamped? Justify your answer. (10)

c) *Extra credit* : Write the solution to this differential equation, in terms of ω_1 where

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2} \qquad (10)$$

Solution : The work done in part a) shows trivially that

$$\omega_1 = \sqrt{25 - 16} = 3$$

and we can write our general solution:

$$x(t) = e^{-\beta t}\left[c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right]$$

using the values for this problem, this becomes

$$x(t) = e^{-4t}\left[c_1 e^{3it} + c_2 e^{-3it}\right]$$

EQUATIONS AND RESULTS

$$\Sigma \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\,\mathbf{v}) = m\,\frac{d\mathbf{v}}{dt} + \mathbf{v}\,\frac{dm}{dt}$$

$$m\,v = m_o\,v_o$$

$$\int \frac{dx}{a\,x + b} = \frac{1}{a}\ln\left|a\,x + b\right| + C$$

$$\int \frac{x}{a\,x^2 + b} = \frac{1}{2\,a}\ln\left|a\,x^2 + b\right| + C$$

$$\int \frac{dx}{a + b\,x^2}\,dx = \frac{\text{ArcTan}\left[\sqrt{b/a}\;x\right]}{\sqrt{a\,b}} + C$$

$$a\,x^2 + b\,x + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$$

$$e^{i\,x} = \cos x + i \sin x$$