PHYS 314
HOMEWORK #2

Due : Feb. 1, 2017

1. We have been discussing the concept of invariance. In special relativity, the quantity that is invariant is :

$$(\Delta x)^2 - c^2 (\Delta t)^2$$

where Δx is the spatial separation of two events and Δt is the temporal separation. If the same events are observed by observers in two different reference frames (call one the primed and the other the unprimed frame), invariance means :

$$(\Delta x)^2 - c^2 (\Delta t)^2 = (\Delta x')^2 - c^2 (\Delta t')^2$$

where the primes refer to the rocket frame and the laboratory is the unprimed frame.

a) You observe two events A and B which occur simultaneously in your lab frame at opposite ends of your lab bench which is 5 m long. Would an observer moving at 0.8 c with respect to your reference frame observe these events to be simultaneous? Use the equations to demonstrate why not? Determine the numerical value of the temporal separation observered in the rocket frame. (Remember to take length contraction into account; you may need to review your notes from modern for this problem).

Solution : If the events occur simultaneously as viewed in the lab frame, we can set $\Delta t = 0$. Our equation then becomes :

$$(\Delta x)^2 = (\Delta x')^2 - c^2 (\Delta t')^2$$

Where $\Delta x = 5$ m is the length of the lab bench in the lab frame, $\Delta x'$ is the length of the lab bench viewed from the rocket frame, and $\Delta t'$ is the time between events viewed in the rocket frame. Since we know that the length of the bench is measured differently in the two frames, we see that $\Delta t' \mathrel{!}= \Delta t$.

In the rocket frame, the lab bench is contracted according to:

$$\Delta x' = \gamma \, \Delta x = \frac{\Delta x}{\sqrt{1 - v^2 / c^2}}$$

so we can rewrite our equation as:

$$(\Delta x)^2 = \gamma^2 (\Delta x)^2 - c^2 (\Delta t')^2$$

This becomes:

$$(\Delta x)^2 \left(\gamma^2 - 1\right) = c^2 (\Delta t')^2 \Rightarrow \Delta t' = \frac{\Delta x}{c} \sqrt{\gamma^2 - 1}$$

For v = 0.8 c, γ = 5/3, so the time between events as measured in the rocket frame is:

$$\Delta t' = \frac{5\,\text{m}\sqrt{16/9}}{3 \times 10^8\,\text{m/s}} = 2.2 \times 10^{-8}\,\text{s}$$

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2. a) Consider the two matrices :

$$A = \begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$$

compute AB and BA. Do these by hand; you may use *Mathematica* to verify your results, but make sure you show your work.

Solution : The ij$^{\text{th}}$ element of the product results from multiplying the i^{th} row of A by the j^{th} row of B :

$$A\,B = \begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 2*3+3*(-1) & 2*1+3*5 \\ -2*3+4*(-1) & -2*1+4*5 \end{pmatrix} = \begin{pmatrix} 3 & 17 \\ -10 & 18 \end{pmatrix}$$

Computing BA :

$$B\,A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3*2+1*(-2) & 3*3+1*4 \\ -1*2+5*(-2) & -1*3+5*4 \end{pmatrix} = \begin{pmatrix} 4 & 13 \\ -12 & 17 \end{pmatrix}$$

b) Consider the matrices :

$$C = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 3 & -1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -2 \\ 4 & 1 & -3 \end{pmatrix}$$

Compute CD and DC by hand. You may use *Mathematica* to verify your results, but show your work in computing these products.

Solution :

$$C\,D = \begin{pmatrix} 1*3+2*1+3*4 & 1*1+2*3+3*1 & 1*(-1)+2*(-2)+3*(-3) \\ -2*3+0+4*(4) & -2*1+0+4*1 & -2(-1)+0+4(-3) \\ 3*3+-1*1+2*4 & 3*1+(-1)*3+2*1 & 3*(-1)+(-1)*(-2)+2*(-3) \end{pmatrix} =$$

$$\begin{pmatrix} 17 & 10 & -14 \\ 10 & 2 & -10 \\ 16 & 2 & -7 \end{pmatrix}$$

Similarly for DC:

$$D\,C = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -2 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 7 & 11 \\ -11 & 4 & 11 \\ -7 & 11 & 10 \end{pmatrix}$$

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3. Find a non-trivial 2 x2 matrix R (i.e., not the identity matrix) such that $R^6 = I$. Show work and/or explain your reasoning.

Solution : The question is asking what matrix could we apply to a vector (or other matrix) six times and return to our original position or value? The rotation matrix of $\theta = \pi/3$ will rotate the axes of coordinates by 60 degrees. If we apply this rotation six times, our axes will wind up in the same configuration we started. The rotation matrix for $\theta = \pi/3$ is :

$$R = \begin{pmatrix} \cos \pi/3 & \sin \pi/3 \\ -\sin \pi/3 & \cos \pi/3 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

We can use *Mathematica* to compute:

In[12]:=
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matrix = {{1/2, Sqrt[3]/2}, {-Sqrt[3]/2, 1/2}};
matrix.matrix.matrix.matrix.matrix.matrix // MatrixForm
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Out[13]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and we obtain the identity matrix as desired.

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4. Verify that the matrix :

$$\begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

is an orthogonal matrix (where the inverse = the transpose).

Solution : If this is an orthogonal matrix, then the transpose is equal to the inverse, thus the product of the transpose and original matrix should yield the identity matrix :

$$R^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and the product :

$$R\,R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since R $R^T = I$, $R^T = R^{-1}$ and this is an orthogonal matrix.

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5. For this problem, refer to Fig. 1 - 4 (a) from the text. Consider the position vector **P** = {1, 2, 3}. Call α, β, γ, the angles between **P** and, respectively, the x, y and z axes. Compute the direction cosines (the cos of the angle between **P** and each of the coordinate axes) and then verify eq. (1.10) from the text.

Solution : Let' s solve this problem generally, using a vector P with coordinates (P_x, P_y, P_z). We can take the dot product between P and each of the unit vectors:

$$\mathbf{P} \cdot \hat{\mathbf{x}}_i = |\mathbf{P}|\,|\hat{\mathbf{x}}_\mathbf{i}| \cos\theta_i = P_i$$

Since the magnitude of all unit vectors is one, we have

$$\cos\theta_i = \frac{P_1}{|\mathbf{P}|}$$

$$\cos\alpha = \frac{P_x}{|\mathbf{P}|};\ \cos\beta = \frac{P_y}{|\mathbf{P}|};\ \cos\gamma = \frac{P_z}{|\mathbf{P}|}$$

Squaring each equation and adding:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{P_x^2 + P_y^2 + P_z^2}{P^2} = 1$$

and we have our desired result.