PHYS 314
HOMEWORK #3

Due : 8 Feb. 2017

1. A uniform chain of mass M, length L and density λ (measured in kg/m) hangs so that its bottom link is just touching a scale. The chain is dropped from rest onto the scale. What does the scale measure at the moment the last link hits the scale?

Solution : This is a variable mass problem, in that we are computing the force exerted by the scale on the chain as the chain falls on the scale. As we discussed in class, part of the force supports the weight of the chain on the scale, and part of the force stops the momentum of the falling chain. We use the general form of Newton' s second law :

$$F = \frac{dp}{dt} = d\frac{(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Since the chain is falling under gravity with no air friction (and no internal friction between the links of the chain), we know that dv/dt = g. Now, we employ the chain rule and obtain:

$$v\frac{dm}{dt} = v\frac{dm}{dy}\frac{dy}{dt}$$

dm/dy is just the mass density of the chain, λ, and dy/dt is the speed of the falling chain. Combining all these, we get :

$$F = m\,g + v^2\lambda$$

where v is the speed of the last link as it hits the scale. We know from elementary physics that an object falling through a height L acquires a speed given by:

$$v^2 = 2\,g\,L$$

so we have:

$$F = m\,g + 2\,g\,L\,\lambda$$

But L λ is just the mass of the entire chain, so we have finally:

$$F = m\,g + 2\,m\,g = 3\,m\,g.$$

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2. Text, problem 11, p. 91

Solution : We have a particle falling in a gravitational field with quadratic air resistance. If we choose down as the negative direction (air friction acts up) and our force equation becomes :

$$F = m\frac{dv}{dt} = -m g + k m v^2$$

(we' ll use the book' s notation since we want to get their answer). Now, the question asks us to find the distance the object falls in accelerating from initial to final velocity, this suggests we should convert dv/dt to dv/dx via :

$$m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m v\frac{dv}{dx}$$

We can separate variables (and divide out a common factor of m) to obtain the differential equation :

$$\frac{v\,dv}{k v^2 - g} = dx$$

Integrating both sides between limits:

$$\int_{v_0}^{v_1}\frac{v\,dv}{k v^2 - g} = \int_{x_1}^{x_2} dx = x_2 - x_1 = -(x_1 - x_2)$$

(Since we chose up to be positive, $x_1 > x_2$.

$$-\frac{1}{2k}\ln\left[k v^2 - g\right]\Big|_{v_0}^{v_1} = s$$

Evaluating between limits and using the properties of logs gives the requested answer:

$$\text{distance traveled} = \frac{1}{2k}\ln\left[\frac{k v_o^2 - g}{k v_1^2 - g}\right]$$

You will obtain the answer in the form given in the book if you set down as the positive direction (and so g would be positive).

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3. Text, problem 12 p. 91

Solution : We break the problem into two parts, the upward portion and the downward portion. For the upward leg, we write :

$$F = m\frac{dv}{dt} = -m g - k m v^2$$

Following the pattern of the problem above, we convert dv/dt to v dv/dy and get:

$$\frac{v\,dv}{g + k v^2} = -dy$$

Integrating yields:

$$\frac{1}{2k}\ln\left[g + k v^2\right] = -y + C$$

Apply the initial condition that $v = v_o$ when y =0:

$$\frac{1}{2k}\ln\left[g + k v_o^2\right] = C$$

and combine results to get:

$$y = \frac{1}{2\,k} \ln\left[\frac{g + k\,v_o^2}{g + k\,v^2}\right]$$

We can use this equation to find an expression for the maximum height of the projectile. Since we know the projectile has zero speed at its highest point, we can write:

$$y_{max} = \frac{1}{2\,k} \ln\left[\frac{g + kv_o^2}{g}\right]$$

Now we consider the downward portion. It is easiest to let down be the positive direction, so that our second law becomes:

$$m\,v\,\frac{dv}{dy} = m\,g - k\,m\,v^2$$

or

$$\frac{v\,dv}{g - k\,v^2} = dy$$

This integrates to:

$$-\frac{1}{2\,k} \ln\left[g - k\,v^2\right] = y + C$$

Since we are choosing down as our positive direction, the trip starts at y = 0 when v = 0, so that we have:

$$\frac{-1}{2\,k} \ln[g] = C$$

which yields :

$$y = \frac{1}{2\,k} \ln\left[\frac{g}{g - k\,v^2}\right]$$

Now, the highest point is common to both the upward and downward portions. This means that their expression for the highest point must be the same, in other words :

$$y_{max} = \frac{1}{2\,k} \ln\left[\frac{g + kv_o^2}{g}\right] = \frac{1}{2\,k} \ln\left[\frac{g}{g - k\,v^2}\right]$$

or

$$\left[\frac{g + kv_o^2}{g}\right] = \left[\frac{g}{g - k\,v^2}\right]$$

Solving for v :

$$v^2 = \frac{g\,kv_o^2}{gk + k^2\,v_o^2}$$

We are asked to express this in terms of the terminal velocity. We can find the terminal velocity from the statement of the second law for the downward leg. Terminal velocity occurs when the velocity is constant, or when:

$$\frac{dv}{dt} = 0 \Rightarrow g - k v^2 = 0 \Rightarrow v_t = \sqrt{\frac{g}{k}}$$

Substituting this into our velocity expression yields the desired result:

$$v = \frac{v_T v_o}{\sqrt{v_T^2 + v_o^2}}$$

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4. Text, problem 36, p. 95.

Solution : We start by writing the equations of motion :

$$x(t) = v_o \cos\theta\, t$$

$$y(t) = h + v_o \sin\theta\, t - \frac{1}{2} g t^2$$

To find the range, we solve for the time of flight by setting y=0, and then use that time in the x(t) equation to find range.

$$y(t) = 0 \Rightarrow \frac{1}{2} g t^2 - v_o \sin\theta\, t - h = 0$$

We use the quadratic equation to find:

$$t = \frac{v_o \sin\theta + \sqrt{v_o^2 \sin^2\theta + 2 g h}}{g} = \frac{v_o}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2 g h}{v_o^2}}\right)$$

Make sure you can explain why we choose the plus branch of the solution. If we substitute this for time in the x(t) equation, we get the range:

$$\text{range} = \frac{v_o^2 \cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2 g h}{v_o^2}}\right)$$

Note that if we set h = 0, we recover the well known range equation for a projectile on level ground with no air friction.

Having taken several semesters of calculus, you know that when you are asked to maximize, you set the first derivative of the appropriate variable to zero. In this case, that means taking the first derivative of range with respect to θ, which gives:

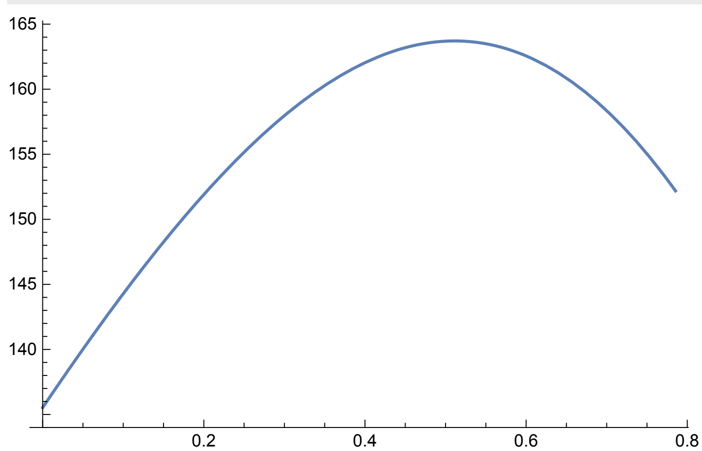
$$\frac{d(\text{range})}{d\theta} =$$

$$\frac{v_o^2}{g}\left[-\sin\theta\left(\sin\theta+\sqrt{\sin^2\theta+\frac{2\,g\,h}{v_o^2}}\right)+\cos\theta\left(\cos\theta+\frac{1}{2\sqrt{\sin^2\theta+\frac{2\,g\,h}{v_o^2}}}\cdot 2\sin\theta\cos\theta\right)\right.=0$$

But this equation is so non-algebraic that is it not possible (or not easily possible) to solve for the value of θ that maximizes range for a certain set of parameters. This is an example of a problem that you have to solve either by numerical or graphical means. Based on recent class work, you might think to expand the radical, but for many values of v_o and h, $2\text{ g }h/v^2$ is not smaller than 1, so that series expansion doesn't help.

Let's see how graphical methods help. We already know that if h = 0 the angle that maximizes range is 45^o. Let's consider:

In[7]:=
```
Clear[range, v0, g, h, θ]
g=9.8;v0=30;h=100;
range=v0^2 Cos[θ](Sin[θ]+Sqrt[Sin[θ]^2 +(2 g h/v0^2)])/g;
Plot[range,{θ,0,π/4}]
```

Out[10]=


Remember that the horizontal axis is in radians, so we can see that the maximum range for these parameters (launch speed = 30 m/s, height = 100m) is about 29^o. We can be more exact by using the FindMaximum command:

In[11]:= **FindMaximum[range, θ]**

Out[11]= {163.711, {θ → 0.511225}}

And we obtain that the maximum range is 163 meters and the launch angle is 0.511 radians (29.3^o). Or we could try to differentiate the range expression, set it equal to zero, and solve for θ by using

FindRoot:

In[13]:= **FindRoot[D[range, θ], {θ, π / 4}]**

Out[13]= $\{\theta \to 0.511225\}$

Technology is your friend.

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5. Suppose the moon is stopped in its orbit at a distance r_o from the Earth and begins to fall inward. Determine the time it will take for the moon to crash into the Earth. Depending on how you approach this problem, you might encounter an integral where the substitution $r = r_o \sin^2\theta$ is useful. Look up the values of the appropriate astronomical parameters (mass of Earth, average distance of moon from Earth, etc.) and calculate the time it will take for the moon to hit the Earth.

Solution : We begin, as all things do, with Newton' a second law :

$$F = m\frac{dv}{dt} = -\frac{G\,m\,M}{r^2}$$

where r is the instantaneous distance of the moon from the Earth. We can get an expression for v in terms of r by setting dv/dt = v dv/dr and:

$$v\frac{dv}{dr} = \frac{-G\,M}{r^2}$$

separate variables, integrate, and:

$$\frac{v^2}{2} = \frac{G\,M}{r} + C$$

we know that v = 0 when r = r_o, so

$$0 = \frac{G\,M}{r_o} + C \Rightarrow C = \frac{-G\,M}{r_o}$$

and :

$$v = \sqrt{2\,G\,M\left(\frac{1}{r} - \frac{1}{r_o}\right)}$$

To find time of infall, we note that v = dr/dt, so:

$$dt = \frac{dr}{v} = \frac{dr}{\sqrt{2\,G\,M}\sqrt{\frac{1}{r} - \frac{1}{r_o}}}$$

Now we make use of our handy substitution:

$$r = r_o \sin^2\theta \Rightarrow dr = 2\,r_o \sin\theta \cos\theta\, d\theta$$

and

$$\sqrt{\frac{1}{r}-\frac{1}{r_o}} = \sqrt{\frac{1}{r_o \sin^2\theta}-\frac{1}{r_o}} = \sqrt{\frac{1-\sin^2\theta}{r_o \sin^2\theta}} = \sqrt{\frac{\cos^2\theta}{r_o \sin^2\theta}}$$

substituting this into our integral:

$$dt = \frac{2\, r_o \sin\theta \cos\theta\, d\theta}{\sqrt{2\,G\,M}\,\sqrt{\frac{\cos^2\theta}{r_o \sin^2\theta}}}$$

or :

$$t = \int dt = \int \frac{2\, r_o^{3/2}}{\sqrt{2\,G\,M}} \sin^2\theta\, d\theta$$

we have left out our limits of integration. In r space our limits are r_o and 0; these translate into θ=0 and $\theta = \pi/2$, so our complete integral is:

$$\sqrt{\frac{2}{G\,M} r_o^3} \int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{\pi}{4}\sqrt{\frac{2}{G\,M} r^3}$$

Substituting values (all in MKS (SI) units):

$$G = 6.67\times 10^{-11};\ M = 6\times 10^{24}\,\text{kg};\ r = 3.84\times 10^{8}\,\text{m}$$

we find that the time to infall is 4.18 10^5s or 4.84 days.