PHYS 314
HOMEWORK #4

Due : 17 Feb. 2017

1. Using the text' s notation, start with equations 2.44 and 2.45 in the text and show all the intermediate steps to derive equations 2.49 and 2.50. (10)

Solution : We use the book' s notation and start with :

$$x = \frac{U}{k}\left(1 - e^{-k t}\right) \qquad\qquad y = \frac{-g t}{k} + \frac{k V + g}{k^2}\left(1 - e^{-k t}\right)$$

where U and V are initial horizontal and vertical velocities respectively. The time of flight, T, is:

$$T = \frac{k V + g}{gk}\left(1 - e^{-k T}\right)$$

Our first step is to expand the exponential in the T equation. This allows us to write:

$$T = \frac{k V + g}{g k}\left(1 - \left(1 - k T + \frac{1}{2} k^2 T^2 - \frac{1}{6} k^3 T^3\right)\right) \Rightarrow T = \frac{k V + g}{g k}\left(k T - \frac{1}{2} k^2 T^2 + \frac{1}{6} k^3 T^3\right)$$

divide through by k on the right:

$$T = \frac{k V + g}{g}\left(T - \frac{1}{2} k T^2 + \frac{1}{6} k^2 T^3\right)$$

Multiply each side by g/(k V+g) and divide by T:

$$\frac{g}{k V + g} = 1 - \frac{1}{2} k T + \frac{1}{6} k^2 T^2$$

Rearrange :

$$\frac{1}{2} k T = 1 - \frac{g}{k V + g} + \frac{1}{6} k^2 T^2 = \frac{k V}{k V + g} + \frac{1}{6} k^2 T^2$$

Multiply through by 2/k and we obtain eq. 2.47 in the text:

$$T = \frac{2 V}{k V + g} + \frac{1}{3} k T^2 = \frac{2 V / g}{1 + k V / g} + \frac{1}{3} k T^2 \qquad (1)$$

Expand the denominator according to:

$$\frac{1}{1 + x} = 1 - x + x^2 \Rightarrow \frac{1}{1 + k V / g} = 1 - k V / g + (k V / g)^2 + \ldots$$

Our equation (1) becomes:

$$T = \frac{2\,V}{g}\left(1 - k\,V/g + k^2\,V^2/g^2\right) + \frac{1}{3}k\,T^2$$

grouping according to k:

$$T = \frac{2\,V}{g} + k\left(\frac{1}{3}T^2 - 2\,\frac{V^2}{g^2}\right)$$

Now, since we have been assuming k is small all along, we can regard the case of small k as a simple perturbation to the case of no friction. This means we can estimate the time of flight (T on the left) by assuming the time on the right can be approximated by the time of flight in the no friction case. The time of flight with no friction is 2 V/g in the book' s notation, so using this for the T^2 term on the right gives:

$$T = \frac{2\,V}{g} + k\left(\frac{1}{3}\cdot\frac{4\,V^2}{g^2} - 2\,\frac{V^2}{g^2}\right) = \frac{2\,V}{g} - \frac{2}{3}\,\frac{k\,V^2}{g^2} = \frac{2\,V}{g}\left(1 - \frac{k\,V}{3\,g}\right)$$

and this is equation 2.50 in the text.

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2. Using the text' s notation, complete the intermediate steps to derive equation 2.55 in the text. (10)

Solution : Now we express the range (R') in the small k case as a perturbation of the range in the no k case. We know the range for a projectile on level ground with no friction is :

$$R = \frac{2\,U\,V}{g}$$

where U and V are again the initial horizontal and vertical velocities. The equation of motion in x is:

$$x = \frac{U}{k}\left(1 - e^{-k\,t}\right) = \frac{U}{k}\left(k\,t - \frac{1}{2}k^2\,t^2 + \frac{1}{6}k^3\,t^3\right)$$

using what are by now familiar series expansion techniques. The flight ends when t = T, so the range is simply x(T) or:

$$R' = U\left(T - \frac{1}{2}k\,T^2\right)$$

Using the value of T from problem 1:

$$R' = U\left[\frac{2\,V}{g}\left(1 - \frac{kV}{3\,g}\right) - \frac{1}{2}k\left(\frac{4\,V^2}{g^2}\left(1 - \frac{2\,k\,V}{3\,g} + \frac{k^2\,V^2}{9\,g^2}\right)\right)\right]$$

Keeping terms out to k:

$$R' = U\left[\frac{2\,V}{g} - \frac{2\,k\,V^2}{3\,g^2} - \frac{1}{2}k\cdot\frac{4\,V^2}{g^2}\right] = \frac{2\,U\,V}{g}\left[1 - \frac{1}{3}\,\frac{k\,V}{g} - \frac{k\,V}{g}\right] = \frac{2\,U\,V}{g}\left(1 - \frac{4}{3}\,\frac{k\,V}{g}\right)$$

or :

$$R' = R\left(1 - \frac{4}{3}\frac{k\,V}{g}\right)$$

which is equation 2.55.

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3. A mass m moves in a circular orbit of radius r_o under the influence of a central force whose potential is -k m / r^n. The total effective potential of the mass is

$$U = \frac{-k\,m}{r^n} + \frac{L^2}{2\,m\,r^2}$$

where L is the constant angular momentum. For what values of n will the circular orbit be stable under small oscillations (that means the mass oscillates about the circular orbit).

Solution : To find stability conditions, we first find the equilibrium conditions, which requires us to find where the first derivative of the potential is zero :

$$\frac{d\,U}{d\,r} = \frac{k\,m\,n}{r^{n+1}} - \frac{L^2}{m\,r^3} = 0 \Rightarrow r^{n+1} = \frac{(m\,r^3)\,k\,m\,n}{L^2}$$

An equilibrium point is stable if the second derivative > 0. Taking the second derivative and evaluating at the equilibrium point:

$$\frac{d^2\,U}{dr^2} = -\frac{(n+1)\,n\,k\,m}{r^{n+2}} + \frac{3\,L^2}{m\,r^4}$$

I will write the r^{n+2} term as r · r^{n+1} so I can use the equilibrium value of r^{n+1}. For stability to exist, we must satisfy the condition:

$$-\frac{(n+1)\,n\,k\,m}{r\,(m\,r^3)\,k\,m\,n} \cdot L^2 + \frac{3\,L^2}{m\,r^4} > 0$$

Cancelling out all common terms leaves only:

$$3 - (n+1) > 0 \Rightarrow n < 2$$

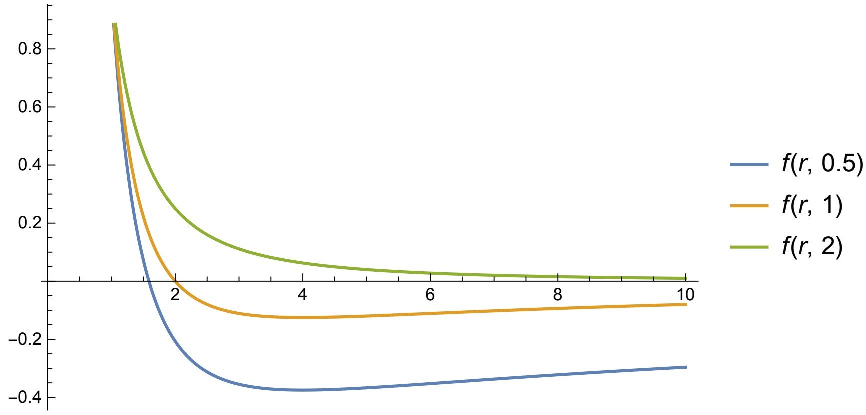
Let' s do some simple plots and see if we can reproduce this result graphically. We' ll choose easy numbers, and investigate the curves of :

$$f = \frac{-1}{r^n} + \frac{2}{r^2}$$

for various values of n. The first graph shows the plot of f for values of n ≤ 2:

In[55]:=
```
Clear[f, r, n]
f[r_, n_] := -1/r^n + 2/r^2
Plot[{f[r, 0.5], f[r, 1], f[r, 2]}, {r, 0.1, 10}, PlotLegends → "Expressions"]
```

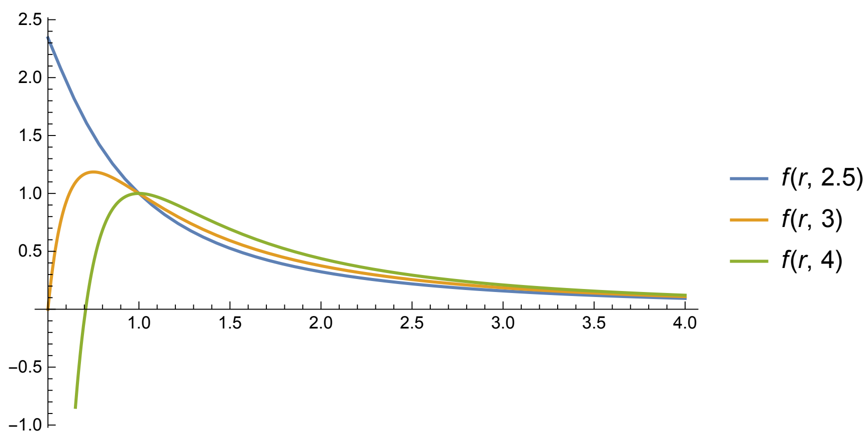
Out[57]=



The next graph shows the function for n>2:

In[23]:=
```
Clear[f, r, n]
f[r_, n_] := -1/r^n + 2/r^2
Plot[{f[r, 2.5], f[r, 3], f[r, 4]}, {r, 0.5, 4}, PlotLegends → "Expressions"]
```

Out[25]=



You can see that the graphs for n <2 have stable minima, but the graphs for n > 2 show unstable extrema.

The next two questions are variable mass problems, the first likely less complex than the second. Please work together on these (if you do, write the names of all group members on your homework; these are problems worth struggling with, so please don't try to find the answers on line or in texts).

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4. A spaceship of surface area A and initial mass m_o and initial speed v_o moves through an interstellar dust cloud of density ρ. As the ship moves through the dust cloud, the dust sticks to the ship and adds to the ship's mass, but exerts no force on the ship.

a) Explain why the rate at which mass accumulates on the ship is (5)

$$\frac{dm}{dt} = \rho A v$$

Solution : In a time dt, the space ship will move a distance v dt and sweep out
a volume of space equal to A v dt. If it encounters dust with a mass density of ρ,
the total amount of dust it encounters in a time dt is just ρ A v dt. Assuming
all the dust sticks to the ship, the amount of mass accreted is simply :

$$dm = \rho v A dt$$

from which our equation derives trivially.

b) Can you use the conservation of energy in this problem? Explain your answer. Is there any conservation law you can use in this problem?(5)

Solution : We cannot. The collisions between the dust and the ship are totally inelastic. We can, however, use conservation of momentum since
there are no external forces acting on the dust / ship system.

c) Find an expression for the velocity of the spaceship as a function of time. (15)

Solution : We use Newton's second law :

$$\frac{d}{dt}(m v) = m \frac{dv}{dt} + v \frac{dm}{dt} = 0 \tag{2}$$

Since there are no external forces acting on the ship, we know its momentum is conserved, which allows us to write:

$$m v = m_o v_o$$

Using these results in equation (2) :

$$\frac{m_o v_o}{v} \frac{dv}{dt} + v (\rho v A) = 0$$

Rewriting :

$$\frac{dv}{v} = -\frac{\rho A v^2}{m_o v_o} dt \Rightarrow \frac{dv}{v^3} = -\frac{\rho A}{m_o v_o} dt$$

Integrating both sides :

$$\frac{-1}{2 v^2} = \frac{-\rho A}{m_0 v_o} t + C$$

Since v = v_o when t = 0, we can evaluate C:

$$C = \frac{-1}{2 v_0^2}$$

and we have:

$$\frac{1}{v^2} = \frac{2\rho A t}{m_o v_o} + \frac{1}{v_o^2}$$

5. A raindrop falling through a cloud accretes matter as it falls. Assume the drop is always spherical and accretes matter according to :

$$\frac{dm}{dt} = \pi r^2 k v$$

where r is the instantaneous radius of the drop, k is a constant, and v is the instantaneous speed of the drop. If the drop starts from rest when it is very small (say r approaching 0), show that the acceleration is constant and equal to g/7. (Note: dr/dt is the rate of increase of the radius of the drop; do not confuse this with the velocity of the drop. You should derive a differential equation which will require you to employ physical reasoning to solve.) (25)

Solution : This is another variable mass problem. In this case, there is an external force, gravity , so we can write Newton' s second law :

$$F = \frac{d}{dt}(m v) = m g$$

or :

$$m \frac{dv}{dt} + v \frac{dm}{dt} = m g \tag{3}$$

and we want to find the acceleration of the raindrop, dv/dt.
First we recognize that since the drop is always spherical, we can write its mass at any time as :

$$m(t) = \frac{4}{3} \pi r^3 \rho \tag{4}$$

where r is the instantaneous radius of the drop and ρ is the density of water. Taking the time derivative of this gives us:

$$\frac{dm}{dt} = 4 \pi r^2 \dot{r} \rho \tag{5}$$

where it is important to remember that r is the radius of the drop, and $\dot{r}$ is the rate at which the radius of the drop grows. Since we have two expressions for dm/dt, we can equate them to find:

$$\pi r^2 k v = 4 \pi r^2 \dot{r} \rho \Rightarrow v = \frac{4\rho}{k} \dot{r} \tag{6}$$

and differentiating one more time :

$$\frac{dv}{dt} = \frac{4\rho}{k} \ddot{r} \tag{7}$$

We rewrite equation (3) as:

$$\frac{dv}{dt} + v \frac{\dot{m}}{m} = g \tag{8}$$

Now, we use the expressions from eq. (7), eq. (6), eq. (5) and eq. (4) in eq. (8) :

$$\frac{4\rho}{k}\ddot{r} + \left(\frac{4\rho}{k}\dot{r}\right)\frac{(4\pi r^2 \dot{r}\rho)}{\frac{4}{3}\pi r^3 \rho} = g$$

Rearranging gives us:.

$$\ddot{r} + \frac{3(\dot{r})^2}{r} = \frac{k}{4\rho} g \qquad (9)$$

Well, you say. This is not too horrible a second order non linear differential equation. I' ll just pop this into Mathematica :

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In[28]:= DSolve[r''[t] + 3 r'[t]^2 / r[t] == c, r[t], t]
(* where c is the constant k g/4 ρ) *)
```

Out[28]= $\left\{\left\{r[t] \to \text{InverseFunction}\left[-\left(\left(\text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{4}{7}, \frac{11}{7}, -\frac{2\,c\,\#1^7}{7\,C[1]}\right] \#1^4 \sqrt{7 + \frac{2\,c\,\#1^7}{C[1]}}\right) \middle/ \left(4\sqrt{7\,C[1] + 2\,c\,\#1^7}\right)\right) \&\right][t + C[2]]\right\},\right.$

$\left.\left\{r[t] \to \text{InverseFunction}\left[\frac{\text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{4}{7}, \frac{11}{7}, -\frac{2\,c\,\#1^7}{7\,C[1]}\right] \#1^4 \sqrt{7 + \frac{2\,c\,\#1^7}{C[1]}}}{4\sqrt{7\,C[1] + 2\,c\,\#1^7}} \&\right][t + C[2]]\right\}\right\}$

And you probably figured that out by just looking. So now, let’s try it as physicists. We have a second order equation which must equal a constant on the right. Well, let’s try a solution of the form:

$$r(t) = c\,t^2$$

where c is some constant. Let’s use this as our trial solution and see what we get:

$$\dot{r} = 2\,c\,t \text{ and } \ddot{r} = 2\,c.$$

Substitute these values into eq. (9) :

$$2c + \frac{3(2ct)^2}{c\,t^2} = \frac{k\,g}{4\rho}$$

$$2c + 12c = \frac{k\,g}{4\rho} \Rightarrow c = \frac{k\,g}{56\rho}$$

And how does this help? Go back to equation (7) and use this value of c to compute dv/dt:

$$\frac{dv}{dt} = \frac{4\rho}{k}\ddot{r} = \frac{4\rho}{k}(2c) = \frac{8\rho}{k}\cdot\frac{k\,g}{56\rho} = g/7$$