# PHYS 314
# HOMEWORK #5

## Due : 24 Feb 2017

1. A 10 kg mass suspended from the end of a vertical spring of negligible mass stretches the mass by 2 cm. Determine the position of the object at any time if it is initially pulled down by 1 cm and then released. Find also the amplitude, period and frequency of motion.

***Solution :*** When the mass is hanging at the end of the spring, we can write Newton' s second law as :

$$m\,\ddot{y} = m\,g - k\,(\Delta y)$$

I am choosing down as the positive direction (so the restoring force is up and negative in this coordinate system). $\Delta y$ is the extension due to the weight of the mass. Since the mass is not accelerating, we know the net force is zero and we can use this information to solve for the spring constant :

$$m\,\ddot{y} = 0 \Rightarrow m\,g = k\,(\Delta y) \Rightarrow k = \frac{10\text{ kg}\left(9.8\text{ m}/\text{s}^2\right)}{0.02\text{ m}} = 4900\text{ N}/\text{m}$$

(this is a very stiff spring). We can also calculate $\omega_o^2$:

$$\omega_o^2 = \frac{k}{m} = \frac{4900\text{ N}/\text{m}}{10\text{ kg}} = 490\text{ s}^{-2} \Rightarrow \omega_o = 22.14\text{ s}^{-1}$$

Next, we are told that the spring is pulled down an additional cm, so that we can write Newton' s second law :

$$m\,\ddot{y} = -\,m\,g + k\,(\Delta y + y)$$

where y is the displacement from the position where the mass hangs. We have already shown that m g = k $\Delta$y, so we can write:

$$m\,\ddot{y} = -k\,y \Rightarrow m\,\ddot{y} + k\,y = 0$$

This is a well known differential equation and we can write its solution as:

$$y\,(t) = A\cos\left(\omega_o\,t\right) + B\sin\left(\omega_o\,t\right)$$

We use our initial conditions to solve for the coefficients A and B. We are told that y(0)= 0.01 m, this condition implies

$$y\,(t = 0) = 0.01\text{ m} = A\cos(0) + B\sin(0) \Rightarrow A = 0.01$$

The second condition tells us that dy/dt (t=0) = 0, this gives us:

$$\frac{dy}{dt} = -\omega_o A \sin(\omega_o t) + \omega_o B \cos(\omega_o t)$$

Evaluating at t = 0:

$$\left.\frac{dy}{dt}\right|_0 = -\omega_o A \sin(0) + \omega_o B \cos(0) = 0 \Rightarrow B = 0$$

Therefore our entire equation of motion is simply:

$$y(t) = 0.01 \cos(\omega_o t)$$

From this equation of motion we can see that the amplitude is 0.01m, the period is $2\pi/\omega_o$ = 0.28s and the frequency is 1/period or 3.57 Hz.

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2. Suppose now the same mass is pulled down 3 cm (instead of 1 cm) and is given an initial velocity of 1 m/s downward. Find the motion at any time, amplitude, period and frequency (and assume the mass is suspended from the same spring as in problem 1).

***Solution*** : The only change from the problem above is in the initial conditions, so we can use the general form of our solution from above :

$$y(t) = A\cos(\omega_o t) + B\sin(\omega_o t)$$

where the period and frequency are unchanged since $\omega_o$ is also unchanged by the changes in the initial conditions. Applying initial conditions as above we find:

$$y(0) = A\cos(0) = 0.03\text{ m} \Rightarrow A = 0.03\text{ m}$$

$$\dot{y}(0) = \omega_o B\cos(0) = 1\text{ m/s} \Rightarrow B = 1/\omega_o = 1\Big/\sqrt{490} = 0.045\text{ m}$$

and our equation of motion becomes:

$$y(t) = 0.03\cos(\omega_o t) + 0.045\sin(\omega_o t)$$

Your book has stated without proof that any combination of sin and cos functions such as:

$$y = A\cos\theta + B\sin\theta$$

can be written as C cos ($\theta$-$\delta$). Read the celestial mechanics classnote for the proof of this (in the section “A Little Trig”). We can write our expression for y(t) as:

$$y(t) = C\cos(\omega_o t - \delta)$$

as long as $C = \sqrt{A^2 + B^2}$ and $\delta = \tan^{-1}(B/A)$.So in our case the amplitude C = 0.054m and $\delta = 56^o$

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3. A particle executing damped harmonic motion obeys the equation :

$$5\ddot{x} + 20\dot{x} + 8x = 0$$

If the particle starts from rest 1 m from the origin:

a) Find the position of the particle at any time, the amplitude and frequency of motion.

Solution : We write this equation in the form of :

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\ddot{x} + 4\dot{x} + 1.6x = 0$$

implies that $\beta = 2$ and $\omega_o^2 = 1.6$

Using results from class and the text, we know we can write our equation of motion as:

$$x(t) = e^{-\beta t}\left[A e^{\sqrt{\beta^2-\omega_0^2}\, t} + B e^{-\sqrt{\beta^2-\omega_0^2}\, t}\right]$$

The condition that x(0)=1 ⇒ A+B = 1.

To simplify our differentiation, let's call $\omega^2 = \beta^2 - \omega_o^2$, then we have:

$$\frac{dx}{dt} = -\beta e^{-\beta t}[A e^{\omega t} + B e^{\omega t}] + e^{-\beta t}[\omega(A e^{\omega t} - B e^{\omega t})]$$

Evaluating at t = 0 :

$$\left.\frac{dx}{dt}\right|_{t=0} = -\beta(A+B) + \omega(A-B) = 0$$

We already know that A+B =1, so:

$$\frac{\beta}{\omega} = A - B$$

For our situation we have that $\beta = 2$ and $\omega_o = \sqrt{1.6} = 1.26$, therefore $\omega = \sqrt{\beta^2 - \omega_o^2} = \sqrt{2.4} = 1.55$

Our simultaneous equations for A and B become:

$$A + B = 1$$

$$A - B = \frac{2}{1.55} = 1.28$$

these combine to :

$$A = 1.14 \text{ and } B = -0.14$$

Our equation of motion is then:

$$x(t) = e^{-2t}\left[1.14\, e^{1.55t} - 0.14\, e^{-1.55t}\right]$$

The "period" of this oscillator is approximated by $2\pi/\omega = 2\pi/1.55/\text{s} = 4\text{s}$. Its motion can be graphed as:

```
Clear[x, β, ω, t]
β = 2; ω = 1.55;
x[t_] := Exp[-β t] (1.14 Exp[ω t] - 0.14 Exp[-ω t])
Plot[x[t], {t, 0, 2 π / ω}]
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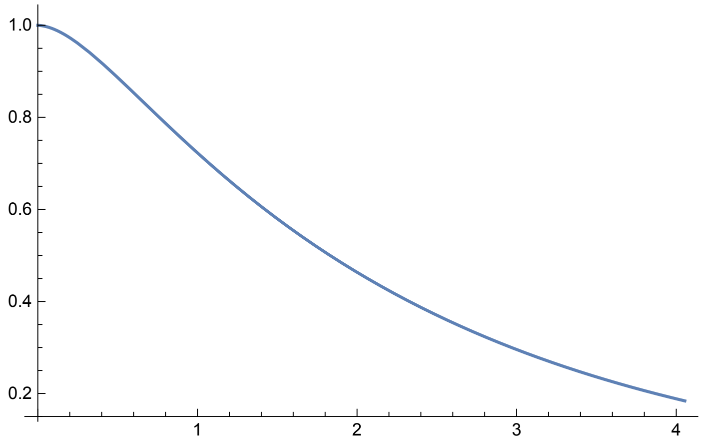

b) Find the logarithmic decrement.

The concept of logarithmic decrement does not apply here since the motion is overdamped.

c) Use Mathematica to plot the motion of this mass and also plot the phase diagram. Submit your Mathematica output with this homework assignment.

The phase diagram plots v[t] vs. x[t]:

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Clear[v, x, β, ω]
β = 2; ω = 1.55;
x[t_] := Exp[-β t] (1.14 Exp[ω t] - 0.14 Exp[- ω t])
v[t_] := -β x[t] + Exp[-β t] (ω (1.14 Exp[ω t] - 0.14 Exp[-ω t]))
ParametricPlot[{x[t], v[t]}, {t, 0, π / ω}]
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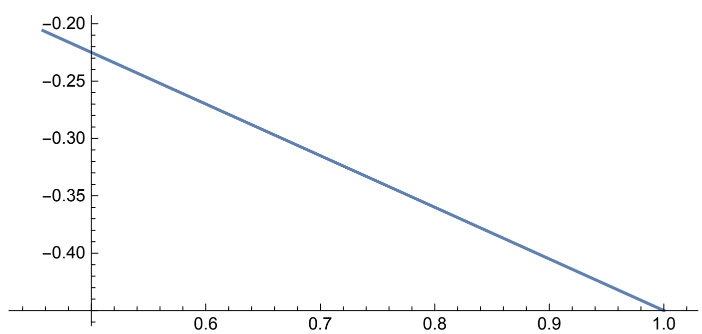

You can review detailed solutions to the next two problems in the link I sent out Wednesday.

4. Text, problem 3 - 8.

5. Text, problem 3 - 12.