PHYS 314
HOMEWORK #7

Solutions

1. Consider the problem, done in class and in the text, of finding the gravitational force due to a spherical shell at points exterior to the shell, interior to the shell, and in the shell. In each case, derive the limits for the dr integral and verify that the text is using the correct values. Verify the integration resulting in eq. 5.21

Solution : Let' s begin with eq. (5.16) in the text :

$$\Phi = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr' \int_{rmin}^{rmax} dr \qquad (1)$$

where r, r' and R are related by the law of cosines:

$$r^2 = r'^2 + R^2 - 2\,r'\,R\cos\theta$$

a) For the case where R>a, θ will vary from 0 to π; when $\theta = 0$, the law of cosines gives us

$$r^2 = (r')^2 + R^2 - 2\,r'\,R = (R - r')^2 \Rightarrow r = R - r'$$

In this case, we know that R > r', so we know that R - r' must be positive and this will be the lower limit of the dr integral. When $\theta = \pi$ we have

$$r^2 = (r')^2 + R^2 - 2\,r'\,R\,(-1) = (r' + R)^2 \Rightarrow r = R + r'$$

This is our upper limit on the dr integral, so that equation (1) becomes:

$$\Phi = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr'\,[R + r' - (R - r')] = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr'\,(2\,r')$$

$$= \frac{-4\pi\rho G}{R}\int_b^a (r')^2\,dr'$$

from which eq. (5.17) follows simply.

b) When R < b, we know that r' > R, so that when $\theta = 0$, r = (r' - R) and r = (r' + R) when $\theta = \pi$. With these limits, we have :

$$\Phi = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr' \int_{rmin}^{rmax} dr = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr' \int_{r'-R}^{r'+R} dr$$

the last integral yields (r’+R)-(r’-R) = 2 R which leads to:

$$\Phi = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr'\int_{rmin}^{rmax} dr = \frac{-2\pi\rho G}{R}\int_b^a r'\,dr'\,(2R) = -2\pi G\rho\left(a^2-b^2\right)$$

which significantly does not bear any R dependence.

c) Finally, when R lies in the shell, we obtain two integrals; the first integral sums contributions to Φ from points b < R < r', and the second integral sums contributions from points r' < R < a, therefore, Φ is the sum of the two integrals :

$$\Phi = \frac{-2\pi\rho G}{R}\left[\int_R^a r'\,dr'\int_{r'-R}^{r'+R} dr + \int_b^R r'\,dr'\int_{R-r'}^{R+r'} dr\right]$$

$$= \frac{-2\pi\rho G}{R}\left[\int_R^a 2Rr'\,dr' + \int_b^R 2r'r'\,dr'\right] = -2\pi\rho G\left(a^2-R^2\right) - \frac{4\pi\rho G}{3R}\left(R^3-b^3\right)$$

and these combine algebraically to eq. 5.21.

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2. Verify eqs. 5.22 in the text.

Solution : Gravity is a conservative force, and therefore is derived by taking the gradient of its appropriate scalar potential. If we take -∇Φ for each of the cases above, we find :

$$\text{a) } g(R > a) = -\frac{d}{dR}\left(\frac{-GM}{R}\right) = -\frac{GM}{R^2}$$

$$\text{b) } g(R < b) = \frac{-d}{dR}\left(-2\pi\rho G\left(a^2-b^2\right)\right) = 0 \text{ since there is no R dependence}$$

$$\text{c) } g(b < R < a) =$$

$$\frac{-d}{dR}\left(-4\pi\rho G\left(\frac{a^2}{2}-\frac{b^3}{3R}-\frac{R^2}{6}\right)\right) = -4\pi\rho G\left(\frac{b^3}{3R^2}-\frac{R}{3}\right) = \frac{-4}{3}\pi\rho G\left(\frac{b^3}{R^2}-\frac{R}{3}\right)$$

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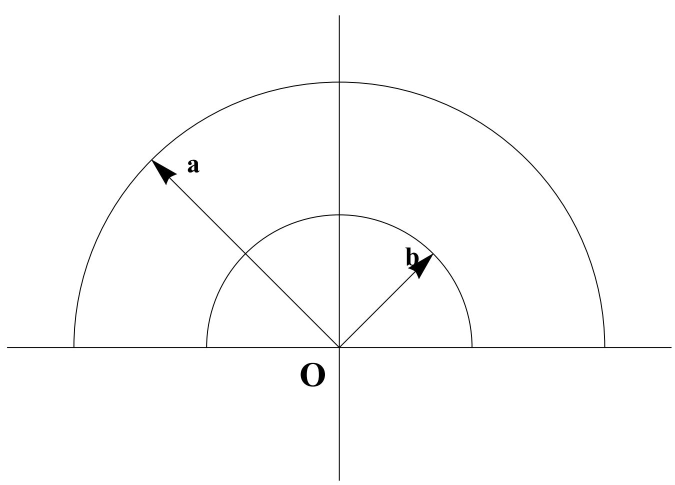
3. A uniform plate has its boundary consisting of two concentric half circles of radii a and b as shown below. Find the force of attraction on a test mass located at the origin (at point O).



Solution : We consider the force of attraction of an element of mass within the concentric shells on a test mass m at the origin. We can write :

$$d\mathbf{F} = \frac{-G\,m\,dM}{r^2}\,\hat{\mathbf{r}}$$

where m is the mass of the test m, dM is the mass of the element of area, and r is the distance between the origin and dM. Since the shell is uniform, we know that dM = ρ dA and in polar coordinates, we can write dA = r dr dθ. (In this problem, we will use r to denote radial distance to avoid confusion with ρ for density.) Our expression for dF becomes :

$$d\mathbf{F} = -\frac{G\,m\,\rho\,r\,dr\,d\theta}{r^2} = -\frac{G\,m\,\rho\,dr\,d\theta}{r}\,\hat{\mathbf{r}}$$

We can use symmetry arguments to show that the x components of the force will cancel, leaving us with only a y component of force. The incremental component of dF in the y direction is:

$$dF_y = -\frac{G\,m\,\rho\,dr\,d\theta\sin\theta}{r}$$

where θ is measured counterclocwise from the positive x axis. Our limits of integration in r and θ give us the following integral :

$$F_y = -G\,m\,\rho\int_b^a\frac{dr}{r}\int_0^\pi\sin\theta\,d\theta = -2\,G\,m\,\rho\int_b^a\frac{dr}{r} = -2\,G\,m\,\rho\ln\left(\frac{a}{b}\right)$$

you can eliminate ρ by writing:

$$\rho = M/A = \frac{M}{\frac{1}{2}\left(a^2-b^2\right)} \Rightarrow F = \frac{-4\,G\,m\,M\ln\left(\frac{a}{b}\right)}{\left(a^2-b^2\right)}$$

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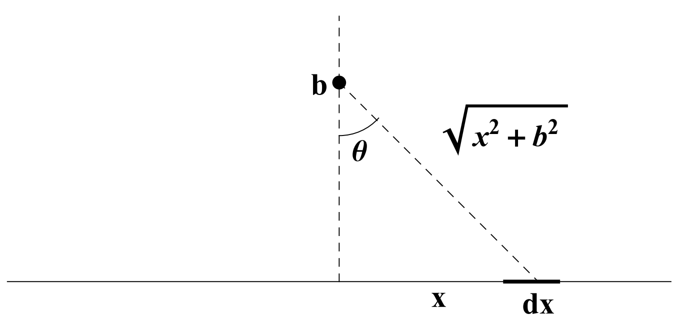
4. Find the force of attraction of a thin uniform rod of length 2 a on a particle of mass m placed at a distance b from its midpoint.

Solution : Let' s begin with the following graphic :



The gravitational force on a test mass at m due to an element of the line denoted by dx is:

$$dF = -\frac{G\,m\,dM}{r^2} = -\frac{G\,m\,dM}{x^2+b^2}$$

where m is the mass of the test particle and dM is the mass of the element dx. If the bar is uniform, its mass can be written as dM = ρ dx and we have:

$$dF = \frac{-G\,m\,\rho\,dx}{x^2 + b^2}$$

We can use symmetry arguments again to show the x components of force will cancel, and the y component of force is given by:

$$dF_y = -\frac{-G\,m\,\rho\,\cos\theta\,dx}{x^2 + b^2}$$

and the geometry of the situation shows us that

$$\cos\theta = \frac{b}{\sqrt{x^2 + b^2}}$$

Thus, the total force at b can be found by integrating from - a to a :

$$F_y = -G\,m\,\rho\,b \int_{-a}^{a} \frac{dx}{\left(x^2 + b^2\right)^{3/2}}$$

and symmetry also allows us to write :

$$F_y = -2\,G\,m\,\rho\,b \int_{0}^{a} \frac{dx}{\left(x^2 + b^2\right)^{3/2}}$$

We can solve this by making the substition x = b tan θ, then

$$x^2 + b^2 = b^2 \tan^2\theta + b^2 = b^2 \sec^2\theta$$

and

$$dx = b \sec^2\theta\, d\theta$$

so that our integral becomes:

$$F_y = -2\,G\,m\,\rho\,b \int_{0}^{\tan^{-1}(a/b)} \frac{b \sec^2\theta\, d\theta}{\left(b^2 \sec^2\theta\right)^{3/2}} = \frac{-2\,G\,m\,\rho}{b} \int_{0}^{\tan^{-1}(a/b)} \cos\theta\, d\theta$$

$$= -\frac{2\,G\,m\,\rho}{b} \sin\theta = \frac{-2\,G\,m\,\rho\,a}{b\sqrt{a^2 + b^2}}$$

and we can eliminate ρ by setting ρ = M/2a so that:

$$F_y = \frac{-G\,m\,M}{b\sqrt{a^2 + b^2}}$$