PHYS 314
HOMEWORK #8

Due : Friday 7 April 2017

1. Starting from the equation describing the element of length :

$$d\mathbf{l} \;=\; h_i\, dq_i\, \hat{\mathbf{q}}_\mathbf{i} \tag{1}$$

where d**l** (also written as d**s**) is the element of length, h represent the scale factors and q represents the spatial coordinates,

a) write d**l** in cylindrical polar coordinates

Solution : Using the scale factors and unit vectors for the cylindrical polar coordeinate system, we have :

$$d\mathbf{l} = d\mathbf{s} \;=\; d\rho\,\hat{\boldsymbol{\rho}} \;+\; \rho\, d\phi\,\hat{\boldsymbol{\phi}} \;+\; dz\,\hat{\mathbf{z}}$$

b) for the specific case of a cone defined by

$$z^2 \;=\; x^2 + y^2$$

show that the scalar element of length can be written as

$$ds \;=\; dz\sqrt{2 + z^2\,(\phi'(z))^2}$$

Solution : Taking **ds· ds** we get :

$$ds^2 \;=\; (d\rho)^2 + \rho^2\,(d\phi)^2 + (dz)^2 \;\Rightarrow\; ds \;=\; \sqrt{(d\rho)^2 + \rho^2\,(d\phi)^2 + (dz)^2}$$

For this specific case, we have:

$$z^2 \;=\; x^2 + y^2 \;=\; \rho^2$$

which impllies that $z = \rho$ and $dz = d\rho$, making these substitutions gives us :

$$ds \;=\; \sqrt{(dz)^2 + z^2\,(d\phi)^2 + (dz)^2} \;\;=$$

$$\sqrt{2\,(dz)^2 + z^2\,(d\phi)^2} \;=\; \sqrt{(dz)^2\left(2 + z^2\left(\frac{d\phi}{dz}\right)^2\right)} \;=\; dz\sqrt{2 + z^2\,(\phi'(z))^2}$$

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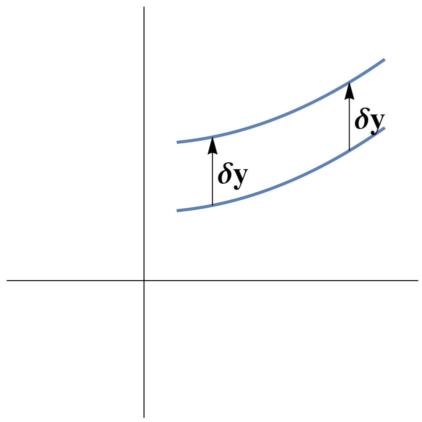
2. Do parts a), b) and c) for problem 14.27 from Felder and Felder (the online chapter on Calculus of Variations). This will complete the proof of why Euler - Lagrange works.

Solution : We will follow the treatment suggested by the book to show that

$$\delta y' = \frac{d}{dx} \delta y$$

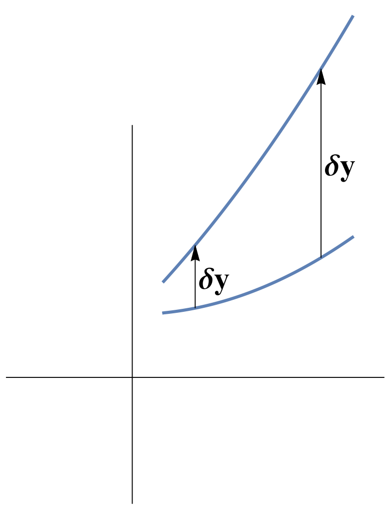
We start by drawing two curves

The upper curve is just the lower curve translated up by one unit. Therefore, the two curves have the same shape and thus the same slope at each point, and δy’ is the same everywhere.



Therefore, the distance between the two curves is the same everywhere, and we have that δy is the same at all points, meaning that d/dx (δy) = 0; the shape of the two curves is the same, the slope of the curve is the same everywhere, so that $\delta y' = 0$ also.

Now, let the two curves vary by an amount that increases as x increases:



Now, we see that δy clearly increases as x increases, so that d/dx (δy) > 0. The slope of the tangent also increases as we go from left to right; since $\delta y'$ is the change in the slope of the line, we can see that $\delta y'$ is related to the rate at which δy changes, so that $\delta y' = d/dx\ (\delta y)$. This completes the proof of the Euler - Lagrange equation.

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3. Start with eq. (1) from above and show that ds on the surface of a sphere of radius r is given by eq. (6.41) in Marion/Thornton.

Solution : For the spherical polar coordinate system, the coordinates are $\{r, \theta, \phi\}$ and the scale factors are $\{1, r, r \sin\theta\}$, so that the element of length in this coordinate system is :

$$ds = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

and taking the dot product gives us:

$$ds^2 = (dr)^2 + r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\theta)^2$$

On the surface of a sphere, r is constant so that dr = 0 leaving us with:

$$(ds)^2 = r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\theta)^2$$

or :

$$ds = r\sqrt{(d\theta)^2 + \sin^2\theta\,(d\phi)^2}$$

If we set f = ds and apply the Euler-Lagrange equation, we will obtain the equaton for the shortest path between two points on a sphere. We use the word geodesic to describe the shortest path on a surface.

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4. Problem 14.51 from Felder and Felder.

Solution : We wish to minimize the integral

$$\int y'^2\,dx$$

subject to y (0) = 0 and y(1)=1

Our function is f= $(y')^2$, so the function that minimizes the integral satisfies:

$$\frac{d}{dx}\left(\frac{\delta f}{\delta y'}\right) - \frac{\delta f}{\delta y} = 0$$

There is no explicit dependence on y, and $\delta f/\delta y' = 2y'$, so we have that

$$2\,\frac{dy}{dx} = c \Rightarrow y = 2\,c\,x + k$$

where c and k are constants. y(0)=0 ⇒ k=0, and y(1)=1 ⇒ c=1/2, so the function that minimizes this integral is y = x/2.

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5. Problem 14.52 from Felder and Felder.

Solution: To minimize

$$\int\sqrt{(y')^2 + x^2}\,dx$$

we set $f = \sqrt{(y')^2 + x^2}$

and solve the Euler-Lagrange equation. First we find:

$$\frac{\delta f}{\delta y'} = \frac{y'}{\sqrt{(y')^2 + x^2}} \quad \text{and} \quad \frac{\delta f}{\delta y} = 0$$

Since there is no explicit y dependence of f, we can set $\delta f/\delta y'$ to a constant and write:

$$\frac{y'}{\sqrt{(y')^2 + x^2}} = c \Rightarrow (y')^2 = c^2 \left((y')^2 + x^2\right)$$

or

$$(y')^2 \left(1 - c^2\right) = c^2 x^2 \Rightarrow \frac{dy}{dx} = k\, x$$

where k is just another constant. This simple differential equation tells us that the curve that minimizes the initial integral is simply

$$y = \frac{1}{2} k\, x^2 + b$$

where b is another constant.

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6. Problem 14.53 from (oh, guess). You may use Mathematica' s DSolve function to solve the resulting ODE, but do the ODEs in the other problems by hand.

Solution : We minimize

$$\int \left(x\, (y')^2 + y^2 / x\right) dx$$

subject to y (1) = 0 and y (2) = 4

Applying Euler - Lagrange, we have :

$$\frac{\delta f}{\delta y'} = 2\, x\, y' \quad \text{and} \quad \frac{\delta f}{\delta y} = 2\, y / x$$

$$\frac{d}{dx}(2\, x\, y') - 2\, y / x = 0$$

The first term is a total derivative, and the equation above becomes:

$$2\, x\, y'' + 2\, y' - 2\, y / x = 0$$

We can solve this using DSolve:

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In[230]:= Clear[y, x]
          DSolve[{2 x y''[x] + 2 y'[x] - 2 y[x] / x == 0, y[1] == 0, y[2] == 4}, y[x], x]
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Out[231]= $\left\{\left\{y[x] \to \frac{8\left(-1 + x^2\right)}{3\, x}\right\}\right\}$

Alternately (and this begins to introduce us to the method of Frobenius), we can assume a solution of the form $y = x^p$ where p is some exponent to be determined. We subsitute this into the differential equation and find:

$$2\,x\,(p\,(p-1))\,x^{p-2} + 2\,p\,x^{p-1} - 2\,x^{p-1} = 0$$

This gives us the equation:

$$[2\,p\,(p-1) + 2\,p - 2]\,x^{p-1} = 0$$

or

$$2\,p\,(p-1) + 2\,p - 2 = 0 \Rightarrow p^2 = 1 \text{ or } p = \pm 1$$

Thus suggests a solution of the form

$$y = a\,x + \frac{b}{x}$$

$$y\,(1) = 0 \Rightarrow a + b = 0 \Rightarrow a = -b$$

$$y\,(2) = 4 \Rightarrow 2\,a - \frac{a}{2} = 4 \Rightarrow a = 8/3 \text{ and } b = -8/3$$

$$\text{thus, } y = 8/3\left(x - \frac{1}{x}\right)$$

and matches the solution from DSolve.