STIRLING' S APPROXIMATION

In class we began to investigate the use of Stirling' s approximation to calculate probabilities of microstates in systems of very large numbers of particles. This short classnote will examine the validity of this approximation.

Let' s start with the more precise form of the approximation, needed when we find factorials of large (merely large) numbers. This form of Stirling' s approximation is :

$$N! \approx N^N e^{-N} \sqrt{2 \pi N}$$

The short program below will show the accuracy of Stirling' s approximation by computing the ratio between the value of N! computed using the approximation to the exact value of N!. We find :

In[9]:=

```
stirling[n_] := n^n Exp[-n] Sqrt[2 π n]
ratio[n_] := stirling[n] / n!
Do[Print["The ratio of the Stirling approximation to the value of n! = ",
  ratio[10^p] // N, "  for n = ", 10^p], {p, 1, 5}]
```

```
The ratio of the Stirling approximation to the value of n! = 0.991704  for n = 10
The ratio of the Stirling approximation to the value of n! = 0.999167  for n = 100
The ratio of the Stirling approximation to the value of n! = 0.999917  for n = 1000
The ratio of the Stirling approximation to the value of n! = 0.999992  for n = 10000
The ratio of the Stirling approximation to the value of n! = 0.999999  for n = 100000
```

We see that this form of Stirling' s approximation is accurate to within 1 % for N as small as 10, and becomes more accurate as N increases. For very large values of N, we can compute the log of N! via :

$$\ln N! = N \ln N - N$$

The program below shows the accuracy of this version of Stirling’s approximation for various values of N:

In[12]:=

```
stirlinga[n_] := n Log[n] - n
ratioa[n_] := stirlinga[n] / Log[n!]
Do[Print["The ratio of the Stirling approximation to the value of ln n! = ",
  ratioa[10^p] // N, "  for n = ", 10^p], {p, 1, 7}]
```

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The ratio of the Stirling approximation to the value of ln n! = 0.862387  for n = 10
The ratio of the Stirling approximation to the value of ln n! = 0.991141  for n = 100
The ratio of the Stirling approximation to the value of ln n! = 0.99926  for n = 1000
The ratio of the Stirling approximation to the value of ln n! = 0.999933  for n = 10000
The ratio of the Stirling approximation to the value of ln n! = 0.999994  for n = 100000
The ratio of the Stirling approximation to the value of ln n! = 0.999999  for n = 1000000
The ratio of the Stirling approximation to the value of ln n! = 1.  for n = 10000000
```

We can see that this form of Stirling' s approx. is not particularly accurate for smaller values of N, but becomes much more accuarate as N increases. Thus, we can be confident of using the more precise version of Stirling for values of N as low as 100, and can use the ln N! = N ln N - N version for values of N greater than a few thousand.