STIRLING' S APPROXIMATION

In class we began to investigate the use of Stirling' s approximation to calculate probabilities of microstates in systems of very large numbers of particles. This short classnote will examine the validity of this approximation.

Let' s start with the more precise form of the approximation, needed when we find factorials of large (merely large) numbers. This form of Stirling’ s approximation is :

\[ N! \approx N^N e^{-N} \sqrt{2\pi N} \]

The short program below will show the accuracy of Stirling' s approximation by computing the ratio between the value of N! computed using the approximation to the exact value of N!. We find:

\[ \text{In[0]} = \] 
\[ \text{stirling[n_]} := n^n \text{Exp[-n]} \text{Sqrt}[2\pi n] \] 
\[ \text{ratio[n_]} := \text{stirling[n]} / n! \] 
\[ \text{Do[Print["The ratio of the Stirling approximation to the value of n! = ", ratio[10^p] // N, ", for n = ", 10^p, \{p, 1, 5\}] } \]

The ratio of the Stirling approximation to the value of n! = 0.991704 for n = 10
The ratio of the Stirling approximation to the value of n! = 0.999167 for n = 100
The ratio of the Stirling approximation to the value of n! = 0.999917 for n = 1000
The ratio of the Stirling approximation to the value of n! = 0.999992 for n = 10000
The ratio of the Stirling approximation to the value of n! = 0.999999 for n = 100000

We see that this form of Stirling’ s approximation is accurate to within 1 % for N as small as 10, and becomes more accurate as N increases. For very large values of N, we can compute the log of N! via :

\[ \ln N! = N \ln N - N \]

The program below shows the accuracy of this version of Stirling’s approximation for various values of N:

\[ \text{In[12]} = \] 
\[ \text{stirlinga[n_]} := n \text{Log[n]} - n \] 
\[ \text{ratioa[n_]} := \text{stirlinga[n]} / \text{Log[n]} \] 
\[ \text{Do[Print["The ratio of the Stirling approximation to the value of ln n! = ", ratioa[10^p] // N, ", for n = ", 10^p, \{p, 1, 7\}] } \]
The ratio of the Stirling approximation to the value of $\ln n!$ is approximately as follows:

- For $n = 10$, the ratio is 0.862387.
- For $n = 100$, the ratio is 0.991141.
- For $n = 1000$, the ratio is 0.99926.
- For $n = 10,000$, the ratio is 0.999933.
- For $n = 100,000$, the ratio is 0.999994.
- For $n = 1,000,000$, the ratio is 0.999999.
- For $n = 10,000,000$, the ratio is 1.

We can see that this form of Stirling's approximation is not particularly accurate for smaller values of $N$, but becomes much more accurate as $N$ increases. Thus, we can be confident of using the more precise version of Stirling for values of $N$ as low as 100, and can use the $\ln N! = N \ln N - N$ version for values of $N$ greater than a few thousand.