SOLUTION FOR IN - CLASS ASSIGNMENT
29 NOV.

Consider a system in thermal and diffusive equilibrium with a reservoir. Show that the average number of particles in the system is expressed as :

$$\overline{N} = \frac{k\,T}{\mathbb{Z}} \frac{\partial \mathbb{Z}}{\partial \mu}$$

where $\mathbb{Z}$ represents the grand partition function defined as :

$$\mathbb{Z} = \Sigma e^{-[E(s) - \mu N(s)]/k\,T}$$

where the summation occurs over all possible s states available to the system.

If we differentiate the grand partition function with respect to μ, we obtain :

$$\frac{\partial \mathbb{Z}}{\partial \mu} = \Sigma \frac{N(s)}{k\,T} \Sigma e^{-[E(s) - \mu N(s)]/k\,T}$$

If we multiply both sides by k T/$\mathbb{Z}$ we get :

$$\frac{k\,T}{\mathbb{Z}} \frac{\partial \mathbb{Z}}{\partial \mu} = \frac{k\,T}{\mathbb{Z}} \Sigma \frac{N(s)}{k\,T} \Sigma e^{-[E(s) - \mu N(s)]/k\,T} = \Sigma N(s) \frac{e^{-[E(s) - \mu N(s)]/k\,T}}{\mathbb{Z}}$$

We can move $\mathbb{Z}$ inside the summation since the grand partition function is the result of summing over all states s, and represents a numerical constant for a given set of values of E (s), μ and T. The ratio of the Gibbs factor to $\mathbb{Z}$ is just the probability of finding the system in a particular state s, so we have that :

$$\frac{k\,T}{\mathbb{Z}} \frac{\partial \mathbb{Z}}{\partial \mu} = \Sigma N(s) \frac{e^{-[E(s) - \mu N(s)]/k\,T}}{\mathbb{Z}} = \Sigma N(s) P(s) = \overline{N}$$

The second part of question 7.6 from the text asks to show :

$$\overline{N^2} = \frac{(k\,T)^2}{\mathbb{Z}} \frac{\partial^2 \mathbb{Z}}{\partial \mu^2}$$

You can proceed as above, taking the second derivative of $\mathbb{Z}$ with respect to μ and manipulating the expressions algebraically to derive the result. Then, for final exam practice, do the next part of the problem that derives an expression for the standard deviation of N.