

PHYS 328

FIRST HOUR EXAM

Fall 2012

This is a closed book, closed note exam. Use of calculators or other electronic devices is not permitted on this test. Do all your writing in your blue book, making sure your name is on each blue book you use. You may do questions in any order, but indicate clearly which question you are solving. All questions must be answered completely clearly showing the work you used to reach your solution. Answers with little to no justification/work will receive little to no credit. Please refer to the sheet of formulae at the end of this test.

1. Consider one mole of an ideal monatomic gas with initial conditions of temperature, volume and pressure given by T_a , V_a , P_a . From this point **a** (on a PV diagram), the gas is isothermally compressed to half its initial volume at point **b**; at point **b**, the gas is compressed at constant volume to a pressure twice its value at point **b**. After the constant volume compression, the gas expands isothermally to its original volume. Finally, the gas expands at constant volume and returns to its initial values of temperature, volume and pressure.

a) Draw a PV diagram showing these stages in the cycle of this gas. (5)

b) Determine the temperature of the gas during its isothermal expansion phase in terms of T_a . (5)

Solution : If we apply the ideal gas law (using the $PV = nRT$ formulation and recalling $n = 1$) to the isothermal compression, we can write for the conditions at point b :

$$P_b V_b = RT_a$$

Since we know that $V_b = 1/2 V_a$ and that the temperature is the same (since the process was isothermal, we can conclude that $P_b = 2 P_a$. The gas is then compressed at constant volume so that the pressure doubles. Thus, calling this point in the PV diagram c, we have that $P_c = 4 P_a$. Applying the ideal gas law to point c gives us:

$$P_c V_c = RT_c \Rightarrow 4P_a (V_a/2) = RT_c \text{ or } 2P_a V_a = RT_c \Rightarrow T_c = 2 T_a$$

b) Determine the total work done in the cycle, making sure your answer includes both magnitude and sign (of course your answer will not be numerical). You should be able to express your answer in terms of temperature (and constants) only. (10)

Solution : The PV work done on or by a gas is :

$$W = - \int_{V_o}^{V_f} P(V) dV \quad (1)$$

Since this process is quasistatic and isothermal, we can express $P(V)$ using the ideal gas law, so the work becomes :

$$W = - \int_{V_0}^{V_f} \frac{RT}{V} dV = - RT \ln\left(\frac{V_f}{V_0}\right)$$

Work is done during the isothermal process, but not during the constant volume phases, so the total work is :

$$W = - RT_a \int_{V_0}^{V_0/2} \frac{dV}{V} - RT_c \int_{V_0/2}^{V_0} \frac{dV}{V} = - RT_a \ln(1/2) - RT_c \ln 2$$

Since $T_c = 2 T_a$ and making use of the properties of logs, our final expression is:

$$W = - RT_a \ln 2$$

The sign is correctly negative indicating the gas does work on its environment; there is more expansion work done than compression, so the net work is negative (work done by the gas on the environment)

c) What is the value of Q for the entire process? (5)

Solution : Since the total change in energy for the process must be zero (the gas returns to its initial state so $\Delta U = 0$), the first law of thermodynamics tells us $Q = -W$ so the total Q for the process must be the negative of the answer in part b).

2. Consider an ideal gas undergoing a quasistatic, adiabatic process.

a) Starting with the basic equations we have used (ideal gas law, 1 st Law of Thermodynamics, equipartition, definition of PV work), derive the relationship (15):

$$P V^\gamma = \text{constant} \quad \text{where } \gamma = (f + 2) / f$$

Solution : This is worked out in detail in the text.

b) One mole of an ideal gas with initial pressure P_0 and volume V_0 is compressed to half its initial volume, how much work is done on the gas during the compression? (5)

Solution : The PV work done by a gas is described by eq. (1) in the first question. In this case, we have :

$$P(V) = \text{constant} * V^{-\gamma} \quad \text{so that } W = - \int_{V_0}^{V_f} \text{constant} * V^{-\gamma} dV = \text{constant} * \frac{V^{1-\gamma}}{\gamma-1} \Big|_{V_0}^{V_f}$$

We can find the value of the constant from the initial conditions :

$$P_0 V_0^\gamma = \text{constant}$$

and we are told that $V_f = V_0/2$. This allows us to write finally:

$$W = \frac{(P_0 V_0)^\gamma}{\gamma - 1} \left[\left(\frac{V_0}{2} \right)^{1-\gamma} - V_0^{1-\gamma} \right]$$

3. Consider a Physics 301 classroom consisting of N students. Use combinatorial analysis to show that the number of different handshakes between two people is given by :

$$\text{no. handshakes} = \frac{1}{2} N(N-1)$$

Solution : We are essentially asking how many different ways can we choose 2 students from an ensemble of N students. Combinatorial analysis yields :

$$\text{no. handshakes} = \binom{N}{2} = \frac{N!}{2!(N-2)!} = \frac{N(N-1)(N-2)!}{2!(N-2)!} = \frac{N(N-1)}{2}$$

(the handshake between A and B counts as 1 handshake, not two separate events). (10)

4. For homework, we showed that the number of microstates available to an Einstein solid of N oscillators with q units of energy at low temperatures is :

$$\Omega(N, q) = \left(\frac{eN}{q} \right)^q \text{ where } q \ll N$$

a) If ϵ is the size of an energy unit, such that $U = \epsilon q$, derive an expression for U in terms of T . (15)

Solution : We will use the definition of temperature :

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)$$

to find a relationship between U and T . First, we use the number of states to find the entropy of this system :

$$S = k \ln \Omega = k \ln \left(\frac{eN}{q} \right)^q = k q (\ln e + \ln N - \ln q) = k \frac{U}{\epsilon} \left(1 + \ln N - \ln \left(\frac{U}{\epsilon} \right) \right)$$

Where we have made use of the information that $U = q \epsilon$. Differentiating this expression for S with respect to U gives :

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k}{\epsilon} \left[\left(1 + \ln N - \ln \left(\frac{u}{\epsilon} \right) \right) - U \cdot \frac{1}{U/\epsilon} \frac{1}{\epsilon} \right] = \frac{k}{\epsilon} \left(\ln N - \ln \left(\frac{U}{\epsilon} \right) \right)$$

Rearranging this last expression gives :

$$\ln\left(\frac{U}{\epsilon}\right) = \ln N - \frac{\epsilon}{kT} \Rightarrow N \epsilon e^{-\epsilon/kT} \quad (2)$$

b) Derive an expression for the heat capacity at constant volume of this system. (10)

Solution : We use the definition of heat capacity :

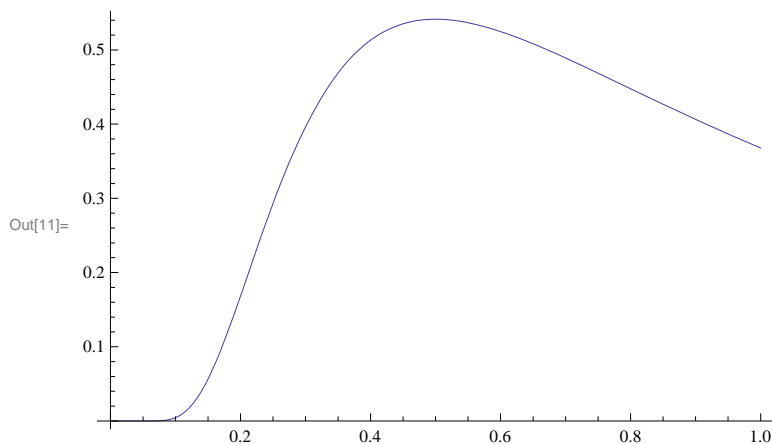
$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

which directs us to take the derivative of equation 2 with respect to T. This yields :

$$C_V = \frac{d}{dT} (N \epsilon e^{-\epsilon/kT}) = N \epsilon e^{-\epsilon/kT} \left(\frac{\epsilon}{kT^2}\right) = \frac{N \epsilon^2}{k T^2} e^{-\epsilon/kT}$$

Note that if we plot this expression for small values of T we see that the heat capacity $\rightarrow 0$ as predicted by the 3rd Law of Thermodynamics :

In[11]:= `Plot[Exp[-1/t]/t^2, {t, 0, 1}]`



5. a) Show that the expression :

$$P = T \left(\frac{\partial S}{\partial V}\right)_{U,N} \quad (3)$$

is dimensionally correct. (The variables have their normal thermodynamic meanings). (5)

Solution : On the left hand side of the equation are the units of pressure which are $N m^{-2}$. On the right, temperature is measured in K (degrees Kelvin), entropy in J/K and volume in m^3 . Thus, the right hand side has units of $J m^{-3}$. However, since a Joule is a $N \cdot m$, the right side becomes:

$$\frac{J}{m^3} \rightarrow \frac{N m}{m^3} \rightarrow N m^{-2}$$

as we have on the left, so the relationship is dimensionally consistent.

b) Consider an ideal gas; use the relationship in part a) to derive the ideal gas law. (15)

Solution: For this part we need to differentiate entropy with respect to volume; to do so, we appeal to the Sackur - Tetrode equation (which you derived for homework) for the entropy of an ideal gas. We start with :

$$S = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right) + 5/2 \right]$$

which looks horrible. But we are only interested in the V dependence of S , and we can write Sackur - Tetrode as :

$$S = Nk \ln V - Nk \ln N + Nk \ln \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} + \frac{5}{2} Nk$$

Then, $\partial S / \partial V = Nk / V$, and we have simply :

$$P = T(Nk / V) \text{ or } PV = NkT$$

The author of the text does a good job of showing how physicists derive equations we already know to give us confidence that the relationship in eq. 3 is correct (since when we apply it to a known situation like an ideal gas) it yields the result we expect.

FORMULAE AND RESULTS

$$P V = n R T$$

$$P V = N k T$$

$$n R = N k$$

$$N = n N_A$$

$$U = N f \cdot \frac{1}{2} k T$$

$$\Delta U = Q + W$$

$$W = F \cdot dr$$

$$W = - \int_{V_i}^{V_f} P(V) dV$$

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V$$

$$C_P = \left(\frac{\partial U}{\partial T} \right)_P + P \left(\frac{\partial V}{\partial T} \right)_P$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

$$\Omega(N, q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!}$$

$$N! = N^N e^{-N} \sqrt{2\pi N}$$

$$\ln N! = N \ln N - N$$

$$\Omega_N = \frac{1}{N!} \frac{v^N}{h^{3N}} \frac{\pi^{3N/2}}{(3N/2)!} (\sqrt{2mU})^{3N}$$

$$\Omega(U, V, N) = f(N) V^N U^{3N/2}$$

$$S = k \ln \Omega$$

$$S = N k \left[\ln \left(\frac{V}{N} \left(\frac{4\pi m U}{3 N h^2} \right)^{3/2} \right) + 5/2 \right]$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)_{N,V}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) \approx x \quad |x| \ll 1$$