SOLUTION FOR EFFICIENCY OF A CARNOT ENGINE

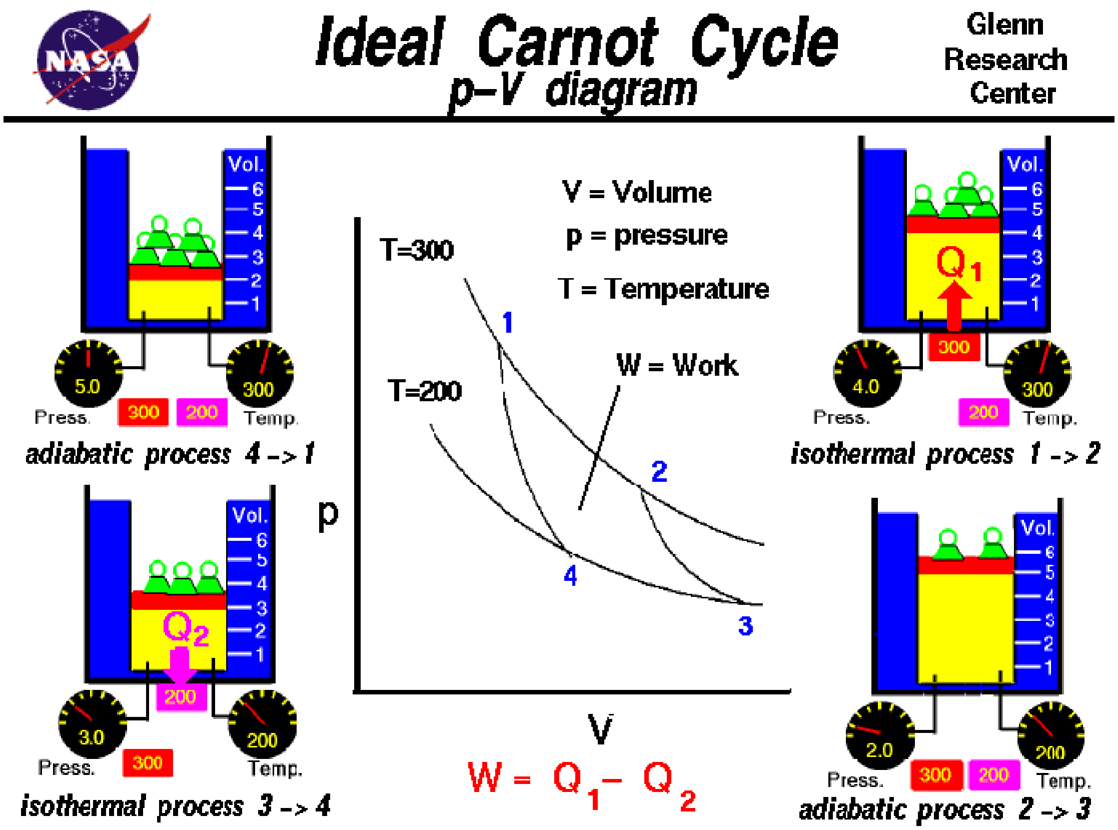
In class, I asked you to prove that the efficiency of a Carnot cycle is :

$$e = \frac{T_h - T_c}{T_h}$$

We will start with the definition of efficiency :

$$e = 1 - \frac{Q_C}{Q_h}$$

where Q_h is the heat extracted from the hot reservoir and Q_C is the heat dumped into the cold reservoir.



We will also use the graphics above (courtesy NASA) to help provide specificity to our calcuation.

The Carnot cycle consists of two isothermal phases (from 1 -> 3 and 3 -> 4) and two adiabatic phases (2 -> 3 and 4 -> 1). No heat is transferred in the adiabatic processes, so the only heat transfer occurs during the isothermal processes. The first law of thermodynamics relates total change in internal energy to Q + W. Since there is no temeprature change, there is no change in internal energy, so the magnitude of Q must equal the magnitude of work. Thus, we can write :

$$Q_h = Q_{12} = |W_{12}| = N k T_h \ln \frac{V_2}{V_1}$$

and :

$$Q_c = Q_{34} = |W_{34}| = N k T_c \ln \frac{V_3}{V_4}$$

Using our definition of efficiency, we obtain :

$$e = 1 - \frac{T_c \ln\left(\frac{V_3}{V_4}\right)}{T_h \ln\left(\frac{V_2}{V_1}\right)}$$

We can readily see that we will obtain our desired solution if we can show the ratio of the ln terms is 1. To do this, we recognize that 2 -> 3 and 4 -> 1 are adiabatic processes, so that along these curves (called adiabats) we have :

$$V T^{f/2} = \text{cst} \Rightarrow V_4 T_c^{f/2} = V_1 T_h^{f/2}$$

also :

$$V_2 T_h^{f/2} = V_3 T_c^{f/2}$$

If you divide these two equations, you will get :

$$\frac{V_3 T_c^{f/2}}{V_4 T_c^{f/2}} = \frac{V_2 T_h^{f/2}}{V_1 T_h^{f/2}} \Rightarrow \frac{V_3}{V_4} = \frac{V_2}{V_1}$$

this final result shows that the ratio of lns above is unity, and we have :

$$e = 1 - \frac{T_c}{T_h}$$

showing that a Carnot cycle of two isothermal and two adiabatic processes achieves the maximum theoretical efficiency.