SOLUTION TO INCLASS ASSIGNMENT (9/24/13)

In class, we used Stirling' s approximation to compute the probability of obtaining exactly 500 heads in a toss of 1000 coins. We solved this for homework 1 using the language one encounters in a basic probability course; today, we solved it using the framework of statistical mechanics. In this framework, we can define the probability of an event as :

$$\text{Probability of H heads} = \frac{\text{multiplicity of H heads}}{\text{total microstates}}$$

In this case, we know the multiplicity of the macrostate of obtaining exactly 500 heads in 1000 flips is :

$$\Omega(1000, 500) = \frac{1000!}{500! \times 500!}$$

and the total number of microstates is simply 2^{1000}. Thus we have:

$$\text{Prob}(1000, 500) = \frac{\Omega(1000, 500)}{2^{1000}} = \frac{\frac{1000!}{500! \times 500!}}{2^{1000}}$$

Using Stirling' s approximation :

$$N! = N^N e^{-N} \sqrt{2\pi N}$$

$$\text{Prob}(1000, 500) = \frac{\frac{1000^{1000} e^{-1000} \sqrt{2\pi(1000)}}{\left(500^{500} e^{-500} \sqrt{2\pi(500)}\right)\left(500^{500} e^{-500} \sqrt{2\pi(500)}\right)}}{2^{1000}}$$

Now, let' s look at this seemingly very complex fraction. We can use the properties of exponents to show that the product of exponents in the denominator is e^{-1000} and cancels the exponent in the numerator. Again using the properties of exponents, we can show that $500^{500} \, x \, 500^{500} = 500^{1000}$ so that we can write:

$$\text{Prob}(1000, 500) = \frac{\frac{1000^{1000}}{500^{1000}} \frac{\sqrt{2\pi(1000)}}{\sqrt{2\pi(500)}\sqrt{2\pi(500)}}}{2^{1000}} = \frac{2^{1000}}{2^{1000}} \sqrt{\frac{2\pi(1000)}{2\pi(500) \cdot 2\pi(500)}} = \sqrt{\frac{1}{500\pi}}$$

The final expression is easily calculated as 0.0252, the probability we determined in the first homework.

Several students asked at the beginning of the assignment if they could ignore the square root factor; as you can see, you will need to keep this term. Look at eq. (1) above. Without the square root terms, you would have predicted a probability of 1, which you know is not correct. Here, we need to use the form of Stirling’s approximation cited at the beginning.