PHYS 328
HOMEWORK #10-- SOLUTIONS

1. Consider the nth energy level in a hydrogen atom. An electron in this level can have n - 1 values of angular momentum, and for each value of angular momentum, can have 2 L + 1 values of the azimuthal quantum number. Thus, the total number of equivalent energy states available to this electron can be expressed as :

$$\sum_{L=0}^{L=n-1} (2\,L+1) \;=\; 2\sum_{L=0}^{n-1} L \;+\; \sum_{L=0}^{n-1} 1$$

Let' s find the value of the last sum; we are simply adding the number one n times (remember, there are n values from 0 to n - 1), so the last sum contributes a value n to the total number of states.

How can we find the value of the first summation; in other words, how do we sum L from 0 to n - 1? You may know that it is well known that the sum of the first k integers is k (k + 1)/2, but let me try to motivate that. Suppose you have integers 1 through k. You can sum them by grouping them in pairs : (1 + k) + (2 + (k - 1)) + (3 + (k - 2)) + ... you can see that the sum of each group is k + 1, and that there are k/2 such pairings, so that the sum of all the integers is (k + 1)×k/2, or k (k + 1)/2. (If there are an odd number of integers, the analysis varies a bit but you get the same result).

Thus the value of the first sum is 2 (n-1) n/2 = n(n-1); when we add this to n we get:

$$\text{total number of energy levels } = n^2 - n + n = n^2$$

You may remember from chemistry that the total number of electrons in an energy level is $2\,n^2$; this takes into account that two electrons of different spin orientations may have the same n, l, m quantum numbers.

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2. Given that :

$$\int \frac{dx}{x^2 + a^2} \;=\; \frac{1}{a}\,\text{arc}\tan\left|\frac{x}{a}\right|$$

we can easily show the definite integral :

$$\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2\,a}$$

Now, if we differentiate both sides with respect to a we get :

$$\frac{d}{da}\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{d}{da}\left(\frac{\pi}{2\,a}\right)$$

The derivative on the right easily yields :

$$\frac{-\pi}{2\,a^2}$$

On the left, we get :

$$\frac{d}{da}\int_0^\infty \frac{dx}{x^2+a^2} = \int_0^\infty \frac{d}{da}\left(\frac{dx}{x^2+a^2}\right) = \int_0^\infty \frac{-2\,a\,dx}{\left(x^2+a^2\right)^2}$$

These two expressions must be equal, so :

$$\int_0^\infty \frac{-2\,a\,dx}{\left(x^2+a^2\right)^2} = \frac{-\pi}{2\,a^2} \;\Rightarrow\; \int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{4\,a^3}$$

Repeating the process we get :

$$\frac{d}{da}\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2} = \frac{d}{da}\left(\frac{\pi}{4\,a^3}\right) = -\frac{3\,\pi}{4\,a^4}$$

The derivative on the left becomes :

$$\frac{d}{da}\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2} = \int_0^\infty \frac{d}{da}\left(\frac{dx}{\left(x^2+a^2\right)^2}\right) = \int_0^\infty -\frac{2\cdot 2\,a}{\left(x^2+a^2\right)^3}\,dx$$

Equating the two expressions :

$$\int_0^\infty -\frac{2\cdot 2\,a}{\left(x^2+a^2\right)^3}\,dx = -\frac{3\,\pi}{4\,a^4} \;\Rightarrow\; \int_0^\infty \frac{dx}{\left(x^2+a^2\right)^3} = \frac{3\,\pi}{16\,a^5}$$

What does Mathematica - bot think?

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Integrate[1 / (x^2 + a^2) ^3, {x, 0, ∞}, Assumptions → Re[a] > 0]
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$$\frac{3\,\pi}{16\,a^5}$$

Efficient, sure, but lacking in the panache of our solution.

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3. In the high temperature limit, the equipartition theorem tells us that the average energy of N oscillators is f/2 N k T. For an Einstein oscillator, f = 2 so that the average energy of the ensemble is N k T. Therefore, the heat capacity is

$$C = \frac{dE}{dT} = N\,k$$

and using the result in the text for the standard deviation of the energy :

$$\sigma_E = k\,T\sqrt{C/k} = k\,T\sqrt{N\,k/k} = k\,T\sqrt{N}$$

The fractional fluctuation in energy is found by dividing the standard deviation by the energy :

$$\frac{\sigma_E}{\bar{E}} = \frac{k\,T\sqrt{N}}{N\,k\,T} = \frac{1}{\sqrt{N}}$$

We have seen previously in the class how $1/\sqrt{N}$ is a good measure of relative fluctuation around the mean. For large systems, i.e., Avogadro sized, $1/\sqrt{N}$ is a tiny number, indicating that any fluctuations around the mean are so small as to be unmeasurable.

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4. We are asked to find an expression for the average energy of a system where the energy is linearly dependent on some coordinate, i.e.,

$$E = c\,|q|$$

where c is some constant. Following the procedure outlined in class, we begin by computing the partition function :

$$Z = \sum_q e^{-\beta c |q|} = \frac{1}{\Delta q} \sum e^{-\beta c |q|}\,\Delta q$$

Allowing Δq to approach zero, we can turn the sum into an integral; making use of the symmetry of | q | we can write :

$$Z = \frac{1}{\Delta q}\int_{-\infty}^{\infty} e^{-\beta c |q|}\,dq = \frac{1}{\Delta q}\cdot 2\int_0^{\infty} e^{-\beta c q}\,dq = \frac{2}{\Delta q}\left(\frac{-1}{\beta c}\right)(0-1) = \frac{2}{\beta c\,\Delta q}$$

Having an analytical expression for Z, we can find the average energy from :

$$\bar{E} = \frac{-1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\beta c\,\Delta q}{2}\left(\frac{2}{c\,\Delta q}\right)\left(\frac{-1}{\beta^2}\right) = \frac{1}{\beta} = k\,T$$

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5. We grind and find using :

$$v_{rms} = \sqrt{3\,k\,T/m}$$

$$v_{max} = \sqrt{2\,k\,T/m}$$

$$\bar{v} = \sqrt{8\,k\,T/(\pi\,m)}$$

Substituting values, we have :

```
Clear[k, temp, mass]
k = 1.38 × 10^-23; temp = 300; mass = 32 × 1.67 × 10^-27;
factor = Sqrt[k temp / mass];
Print["The rms speed = ", Sqrt[3] factor, " m/s"]
Print["The most probable speed = ", Sqrt[2] factor, " m/s"]
Print["The average speed = ", Sqrt[8 / π] factor, " m/s"]
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The rms speed = 482.089 m/s

The most probable speed = 393.624 m/s

The average speed = 444.158 m/s
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6. Following the treatment in the text, we find the most probable velocity of a nitrogen molecule at this temperature :

$$v_{max} = \sqrt{2\,k\,T/m} = \sqrt{2 \cdot 1.38^{-23}\,J/K \cdot 1000\,K \Big/ \left(28 \cdot 1.67 \times 10^{-27}\,kg\right)} = 768\,m/s$$

The required speed for escape, 11, 000 m/s is 14.3 times this, so we can use text eq. 6.55 with x_{min} = 14.3 to compute:

$$P\,(v > v_{exc}) = \frac{4}{\sqrt{\pi}} \int_{14.3}^{\infty} x^2\, e^{-x^2}\, dx$$

Integrating numerically gives :

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(4 / Sqrt[π]) NIntegrate[x^2 Exp[-x^2], {x, 14.3, ∞}]
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2.51172×10^{-88}

Which is really small. Realizing that the lifetime of the Earth is approx. 4.5 billion years (or 10^{17}secs), it is highly unlikely that the Earth will lose much nitrogen over geological history. However, for hydrogen and helium, the masses are much smaller and:

$$v_{max}\,(H_2) = 2874\,m/s$$

so that x_{min} = 11,000/2874 = 3.83 and:

$$P\,(v > v_{esc}) = \frac{4}{\sqrt{\pi}} \int_{3.83}^{\infty} x^2\, e^{-x^2}\, dx$$

```
Print["The probability of H₂ escape = ",
 (4 / Sqrt[π]) NIntegrate[x^2 Exp[-x^2], {x, 3.83, ∞}]]
```

The probability of H_2 escape = 1.9017×10^{-6}

While a small number, we should not be surprised that over the lifetime of the Earth, hydrogen molecules have essentially all escaped, so that there is no free H left in the Earth' s atmosphere. A similar result will follow for Helium which is only twice as massive as hydrogen molecules (where x_{min} will be $\sqrt{2}$ times larger for He than H_2

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Print["The probability of He escape = ",
 (4 / Sqrt[π]) NIntegrate[x^2 Exp[-x^2], {x, Sqrt[2] 3.83, ∞}]]
```

The probability of He escape = 1.12757×10^{-12}

Since the moon and Earth are roughly the same distance from the sun, we should expect the same temperatures in the planet' s exospheres. However, on the moon the escape velocity is much lower (2400 m/s), so the value of x_{min} for nitrogen on the moon is 2400/768 = 3.12, leading to a probability of exceeding the escape velocity of:

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(4 / Sqrt[π]) NIntegrate[x^2 Exp[-x^2], {x, 3.12, ∞}]
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0.000218681

or $2.2 \cdot 10^{-4}$, suggesting a high probability of escape, consistent with the observation that the moon has no atmosphere today.