PHYS 328

Homework #1--Solutions

1. This question explores the relative cost of human vs. fossil fuel energy. A human being hard at work will expend approximately 100 W of power (if you use an exercise bike or treadmill that displays power, see how hard you have to work to expend 100 W). A small compact car traveling at highway speeds will expend approximately 100 kW. Use available data (such as the federal rate for reimbursing mileage) to determine how much money is required to operate a car for an hour. Now, suppose that the energy is provided by work study students pushing the car. How many students would you need to move the car at the same speed? Using the current U.S. minimum wage, how much would it cost to operate that car for an hour? Show all work and state explicitly all assumptions you make and values you use in your calculations.

Solution : The current federal mileage reimbursement rate is 55.5 cents/mile (http://www.irs.gov/newsroom/article/0,,id=250882,00.html); at highway speeds of say 55 mi/hr, the cost of operating a car for 1 hour is then 55.5 cents/mile x 55 mi = $30 .52.

Since the power generated by the internal combustion engine of the car is 1000 times greater than typical human power output, we would require the services of 1000 work study students to propel the car at the same speed. The current federal minimum wage is $7.25/hr, but the Illinois minimum wage is $8.25/hr. Using human power would cost $8250 in Illinois. Gas seems relatively cheap now, doesn't it?

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2. In our study of thermodynamics, we will frequently compute partial derivatives of functions of several variables. This question illustrates the importance of being mindful which variable (s) is being held constant. Consider two functions, w = x y and x = y z.

a) Write w purely in terms of x and z, and then purely in terms of y and z.

b) Compute the partial derivatives :

$$\left(\frac{\partial w}{\partial x}\right)_y \text{ and } \left(\frac{\partial w}{\partial x}\right)_z$$

where the subscript indicates the variable that is held constant. Show that these two partial derivatives are not equal.

Solution : Start by writing:

$$x = yz \Rightarrow y = \frac{x}{z} \Rightarrow w(x, z) = x\left(\frac{x}{z}\right) = \frac{x^2}{z};\ \left(\frac{\partial w}{\partial x}\right)_z = \frac{2x}{z}$$

$$w = x\,y \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y = y. \quad \text{Since } y = \frac{x}{z},$$

we can see the two results are not equivalent for all values of x.

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3. On a clear autumn day in the midlatitudes, it is possible for the evening low to reach 30^o F and the afternoon high to reach 60^o F. Is it meaningful to say that it was twice as hot at the afternoon high than at the evening low? If so, explain why; if not, explain why not.

Solution : It is not a meaningful statement because the zero point of the Fahrenheit scale is arbitrarily chosen. In order to make meaningful relative comparisons between two temperatures, we must use the Kelvin scale since in that scale, 0 K represents a "true zero".

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4. Determine the number of molecules in a cubic meter of air at the surface of the Earth at room temperature and 1 atm of pressure.

Solution : We begin with the perfect gas law:

$$PV = N\,k\,T \Rightarrow N = \frac{P\,V}{k\,T} = \frac{10^5\,N\,m^{-2} * 1\,m^3}{1.38\,x10^{-23}\,J/K * 300\,K} = 2.4 \times 10^{25}\text{ molecules}$$

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5. The number density of the atmosphere decreases exponentially with height according to :

$$n(z) = n_0\,e^{-z/H}$$

where n (z) is the number density at any height z above the surface of the Earth, n_o is the number density at the surface of the earth, and H is a constant called the scale height of the atmosphere. For the Earth, the scale height is 8 km.

Find the total number of molecules in a 1 m^2 column reaching from the surface of the Earth to the “top” of the atmosphere, several hundred kilometers above the surface of the Earth. Assuming the Earth is a sphere of radius R (R = 6400 km), estimate the total number of molecules in the Earth' s atmosphere.

Solution : We start by considering how many molecules are in a volume dV, given by:

$$dV = A\,dz$$

where A is the cross sectional area of the region and dz is the height of the region. We can set A to be a constant 1 m^2, so that the total number of molecules in this region is:

$$N(z) = n(z)\,dV = n(z)\,dz = n_0\,e^{-z/H}\,dz$$

Then, the total number of molecules in a column of this cross - sectional area is :

$$N_{total} = \int_0^{\text{top of atmosphere}} N(z)\,dz$$

Integrating simple exponential functions is easy, but what do we choose for our upper limit, since I did not give you a specific value for the "top of the atmosphere"? Let' s see if we can justify the use of infinity for our upper limit, because if we can, we get the particularly nice result :

$$N_{total} = H\, n_0$$

and we already know the value of n_0 from problem 4. How can we justify using ∞ as our upper limit since we know the atmosphere isn' t really infinite in extent? But we do know that the atmosphere extends to at least 80 km (which represents 10 scale heights), so we can estimate our fractional error in assuming an upper limit of infinity by calculating :

$$\int_{10}^{\infty} e^{-x}\, dx = e^{-10} = 4.5 \times 10^{-5}$$

$$\left(\text{Some quick rules of thumb : } e^{-3} \approx \frac{1}{20};\ e^{-5} \approx \frac{1}{150}\right)$$

We can see that the error associated with using an upper limit of infinity is miniscule compared to other assumptions we have made (that I didn' t tell you about, mostly assuming an isothermal atmosphere to derive our exponential law) so that we can easily calculate that the total number of molecules in a column 1 m^2 in area is :

$$N_{total} = H\, n_0$$

and that the total number of molecules in the atmosphere is simply :

$$4\,\pi\, R^2\ H\, n_0 = 4\,\pi \left(6.4 \times 10^6\ \text{m}\right)^2 (8000\ \text{m}) \left(2.4 \times 10^{25}\ \text{molecules} / \text{m}^3\right) \approx 10^{44}\ \text{molecules}$$