

PHYS 328

HOMEWORK #2--Solutions

1. Estimate how many air molecules you inhale in each breath. If you never inhale the same air molecule twice, how long would you have to live to inhale every molecule in the atmosphere? Explain all estimates you make and/or cite all references for values you use in solving this problem.

Solution : During normal deep breathing, I will inhale approximately 2 liters of air per breath, and I take approximately 16 breaths per minute, or 0.267 breaths/s. We calculated the density of air on the surface of the Earth in the first homework assignment, so we can conclude :

$$\text{Molecules inhaled / s} = (\text{breaths / s}) (\text{molecules / m}^3) (\text{m}^3 \text{ inhaled / breath})$$

$$\begin{aligned} \text{Molecules inhaled / s} &= (0.267 \text{ breath / s}) (2.4 \times 10^{25} \text{ molecules / m}^3) (0.002 \text{ m}^3 / \text{breath}) \\ &= 1.3 \times 10^{22} \text{ mol / s} \end{aligned}$$

Then, the length of time it would take to inhale all 10^{44} molecules in the atmosphere is $\sim 10^{44} \text{ molecules} / 10^{22} \text{ molecules/s} \approx 10^{22} \text{ s}$. The age of the universe is approximately $5 \times 10^{17} \text{ s}$. So...unlikely.

2. Problem 1.16 from the text, page 8, parts a) - c) only. (5 pts for each part)

Solutions :

a) If the slab is not accelerating, we know all the forces acting on it sum to zero. We will apply Newton's second law to the atmospheric slab. In doing so, we recall that forces acting on fluids (i.e., gases and liquids) are transmitted equally in all directions (Pascal's Law). Therefore, the atmosphere exerts a pressure downward on the upper slab of the atmosphere, and upwards on the lower slab of the atmosphere. If the thickness of the atmosphere is dz , then the downward pressure at the top is $P(z + dz)$ and the upward pressure on the bottom is $P(z)$.

If the density of the slab is ρ , the total mass of the slab is $\rho A dz$ where A is the cross sectional area of the slab, thus its weight is $\rho g A dz$ where g is the acceleration due to gravity.

Recalling that force and pressure are related by $F = PA$, we write Newton's second law for the slab:

$$P(z + dz) A + \rho g A dz = P(z) A$$

The A terms cancel; rearranging yields :

$$\frac{P(z + dz) - P(z)}{dz} = -\rho g$$

In the limit that $dz \rightarrow 0$, the term on the left is just dP , and we have the **equation of hydrostatic**

equilibrium :

$$\frac{dP}{dz} = -\rho g$$

b and c) Combining the equation of hydrostatic equilibrium and the ideal gas law in the form :

$$P = \frac{\rho k T}{\mu m_H} \Rightarrow \rho = \frac{\mu m_H P}{k T} \Rightarrow \frac{dP}{dz} = -\frac{\mu m_H g P}{k T}$$

Separation of variables immediately yields the almost trivial differential equation :

$$\frac{dP}{P} = -\frac{\mu m_H g}{k T} dz$$

Which has the solution :

$$P(z) = P(0) \text{Exp}\left[-\frac{\mu m_H g z}{k T}\right]$$

where $P(0)$ is the pressure at the surface of the Earth. Let's take a deeper look at this expression. We know that the argument of an exponential must be dimensionless, so we know that the expression :

$$\frac{\mu m_H g}{k T}$$

must have the units of 1/distance, so that the expression :

$$\frac{k T}{\mu m_H g}$$

will have units of distance. This latter expression is the equation for the scale height that we introduced in homework set 1. Evaluating this using terrestrial parameters we find :

$$H = \frac{k T}{\mu m_H g} = \frac{(1.38 \times 10^{-23} \text{ J/K})(273 \text{ K})}{29(1.67 \times 10^{-27} \text{ kg})(9.8 \text{ m/s}^2)} = 8000 \text{ m}$$

where we use a mean surface temperature of 273 K. With this definition of H , we can write hydrostatic equilibrium in the form :

$$P(z) = P(0) e^{-z/H}$$

3. Calculate the rms speed of an oxygen molecule at the surface of the Earth (i.e., room temperature).

Solution : The relevant equation to use here is :

$$v_{\text{rms}} = \sqrt{\frac{3 k T}{m}}$$

The molecular weight of diatomic oxygen is 32 (the mass of one proton is 1.6×10^{-27} kg), so we compute:

$$v_{\text{rms}} = \sqrt{\frac{3 k T}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \text{ J/K } 300 \text{ K}}{32 \times 1.6 \times 10^{-27} \text{ kg}}} = 492 \text{ m/s}$$

4. Problem 1.22 from the text, page 14, parts a) - c) and either d) or e). (5 pts each part)

Solutions :

a) Following the discussion in the text and classnotes, we have shown that the average pressure exerted by a single molecule on a wall is :

$$P = - \frac{m \Delta v_x}{A \Delta t}$$

For a molecule rebounding from a wall with no energy lost on collision, $\Delta v_x = - 2 v_x$. If there are a total of N molecules, then we can write the pressure due to collisions on the wall as:

$$P = \frac{2 N m \Delta v_x}{A \Delta t} \Rightarrow N = \frac{P A \Delta t}{2 m \Delta v_x}$$

Note that the expression for N tells us how many molecules will strike a section of wall of area A in a time Δt .

b) In class notes and in the text, we show that :

$$P V = N k T = N m \overline{v_x^2} \Rightarrow m \overline{v_x^2} = k T$$

or that :

$$\overline{v_x^2} = \sqrt{k T / m}$$

c) If we now imagine there is a hole of area A in the wall, the expression we calculated for N in part a) becomes the number of molecules that will leave the chamber; in essence, the expression for N in part a) is our equation for ΔN , and we have :

$$\Delta N = - \frac{P A \Delta t}{2 m \Delta v_x}$$

Using the ideal gas law to rewrite P, and using our expression for v_x derived in part b), we can write:

$$\Delta N = - \frac{\left(\frac{NkT}{V}\right) A \Delta t}{2 m \sqrt{kT/m}} = - \frac{A \Delta t}{2 V} \sqrt{kT/m} N$$

In the limit that $\Delta t \rightarrow 0$, we can write the above equation in differentials and separate variables :

$$\frac{dN}{N} = - \frac{A}{2 V} \sqrt{kT/m} dt$$

Employing the technique we used in problem 2 to define the scale height (or e - folding height), let's define the e - folding time (or characteristic time or relaxation time) as :

$$\tau = \left(\frac{A}{2 V} \sqrt{kT/m} \right)^{-1}$$

so that our differential equation becomes :

$$\frac{dN}{N} = - dt / \tau \Rightarrow N(t) = N(0) e^{-t/\tau}$$

where $N(0)$ is the number of molecules in the chamber at $t = 0$. The expression for $N(t)$ tells us how many molecules will reside in the chamber as a function of time.

d) The trick here is to be consistent with units. The characteristic time is :

$$\tau = \frac{2 V}{A} \sqrt{m/kT}$$

and our values are :

$$V = 1 \text{ L} = 10^{-3} \text{ m}^3$$

$$A = 1 \text{ mm}^2 = 10^{-6} \text{ m}^2$$

$$m = 29 \times 1.67 \times 10^{-27} \text{ kg}$$

$$k = 1.38 \times 10^{-23} \text{ J/K}$$

$$T = 300 \text{ K}$$

Plug and chug and obtain :

$$\tau = 1.27 \text{ s}$$

In other words, for each 1.27 s period that elapses, the density in the chamber will decrease by a factor of e. In 5 characteristic times (i.e., 6.35 s), the number of molecules in the chamber will be e^{-5} or $\sim 1/150$ of the initial abundance. In 10 characteristic times, or 12.7s, the number of molecules will be e^{-10} or 1/22,500 of the initial amount.

e) For this part, we will calculate the area of a hole that will cause a tire to deflate in an hour. Thus,

we will solve for area in the characteristic time equation :

$$A = \frac{2V}{\tau} \sqrt{m/kT}$$

We can use 1 hour (3600 s) as the characteristic time; we can use the same values for m and T as we did in part d). The only remaining variable to estimate is the volume of the tire. Imagine the tire unraveled from the wheel, so it forms a cylinder whose length is the circumference of the tire. A typical bicycle tire is 28 inches in diameter, or 0.7m for a circumference of approx 2.2m. A typical thickness of a bicycle tire is an inch or two (2 - 4 cm, so I will choose 3 cm for grins). Thus, the volume of a bicycle tire is:

$$V = \pi r^2 h = 3.14 (0.03 \text{ m})^2 (2.2 \text{ m}) = 0.006 \text{ m}^3 \text{ (or 6 L)}.$$

Substituting values, we obtain :

$$A = \frac{2(0.006 \text{ m}^3)}{3600 \text{ s}} \sqrt{29 \times 1.67 \times 10^{-27} \text{ kg} / (1.38 \times 10^{-23} \text{ J/K} * 300 \text{ K})} = 1.1 \times 10^{-8} \text{ m}^2$$

or a radius of $6 \times 10^{-5} \text{ m}$.

5. Problem 1.24 from the text, page 17.

Solution : We make use of the equipartition function and write that the total thermal energy in a system of N particles is :

$$U_{\text{th}} = N f \left(\frac{1}{2} k T \right)$$

We are told the temperature and we will determine N knowing our sample has a mass of 1 g. We need to determine f, the number of degrees of freedom. Since the sample is solid lead, we expect the atoms to be bound in a lattice structure. Refer to Fig. 1 - 6 on page 16 of the text; this model of crystalline solids suggests that the only degrees of freedom are vibrational (which we assume are not frozen out). An atom can vibrate in any one of three coordinates, and each coordinate has two degrees of freedom associated with it. Thus, there are 6 degrees of freedom for atoms in this solid. The atomic mass of lead is 207.2, so our sample is 1/207.2 moles of lead. Therefore, the number of lead atoms in this sample is :

$$N = 6 \times 10^{23} \text{ atoms/mole} * \frac{1}{207.2} \text{ moles}$$

and the total thermal energy is then :

$$U_{\text{th}} = 2.9 \times 10^{21} * 6 * \left(\frac{1}{2} * 1.38 \times 10^{-23} \text{ J/K} * 300 \text{ K} \right) = 36 \text{ J}$$