PHYS 328
HOMEWORK #3

Due : 13 Sep. 2012

1. Text, problem 1.28

Solution : Let'**s** assume the water is initially at 20 C (approx room temperature), and that a cup of water is defined as approx 200-284 ml of liquid. Let'**s** choose 250 ml to be around the midpoint of this range. We need to calculate how long it will take to heat this water to 100C if our energy input is 600J/s. We know that we require 4.2J of energy to raise the temp of 1 g of water by 1^0C, so the total energy required to warm this much water to 100^0C is:

$$\text{Energy} = 250\,\text{g} \ast 4.2\,\text{J}/\text{g} \ast (100 - 20)^0\,\text{C} = 84,000\,\text{J}$$

If the microwave generates 600 J of energy /sec, the time required is :

$$\text{time} = 84,000\,\text{J}/600\,\text{J}/\text{s} = 140\,\text{s}$$

Heat describes the flow of energy between a hotter and colder object in contact, a situation that does not exist in this case. The energy transfer here is due to microwave radiation, and is an example of work.

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2. Text, problem 1.29

Solution : The trick is that you cannot make any meaningful comments about theamount of heat added to the system. We know that the energy of the system increased, but have no information telling us how much of that internal energy increase was due to heat or due to work (say, someone picked up the cup and shook it so vigorously that the temperature increased). The question is intended to draw the distinction between heat and work.

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3. Text, problem 1.34, all parts. Five pts per part.

Solutions : For this problem, we will make frequent use of the relations:

$PV = NkT$ (Eq. A)

$U_{th} = \dfrac{f}{2} NkT = \dfrac{f}{2} PV$ (Eq. B)

$\Delta U = Q + W$ (Eq. C)

$W = - \displaystyle\int P\,dV$ (Eq. D)

Step A : The volume is fixed and heat is added to increase the pressure. Since the volume is fixed, the work done is zero. The change in thermal energy is given by :

$$\Delta U_A = \frac{f}{2} \Delta (PV) = \frac{f}{2} V_1 (P_2 - P_1)$$

The first law of thermodynamics tells us that this is also the heat input for step A.

Step B : Here, the gas expands at constant pressure so the work done is negative (since the gas is doing work on the environment). The amount of work is :

$$W_B = - P_2 (V_2 - V_1)$$

The change in thermal energy is :

$$\Delta U_B = \frac{f}{2} \Delta (PV) = \frac{f}{2} P_2 (V_2 - V_1)$$

The first law of thermo (Eq. C) tells us that Q for Step B is :

$$Q_B = \Delta U - W = \frac{f}{2} P_2 (V_2 - V_1) + P_2 (V_2 - V_1) = \left(\frac{f + 2}{2}\right) P_2 (V_2 - V_1)$$

Step C : The work is zero since ΔV is zero. Thus, $\Delta U = Q$, and we find ΔU from:

$$\Delta U_C = Q_C = \frac{f}{2} \Delta (PV) = \frac{f}{2} V_2 (P_1 - P_2) = \frac{-f}{2} V_2 (P_2 - P_1)$$

Step D : Here the pressure is held constant and positive work is done on the gas to compress it. We have that:

$$W_D = - P_1 (V_1 - V_2) = P_1 (V_2 - V_1)$$

$$\Delta U_D = \frac{f}{2} \Delta (PV) = \frac{f}{2} P_1 (V_1 - V_2) = \frac{-f}{2} P_1 (V_2 - V_1)$$

$$Q_D = \Delta U_D - W_D = \frac{-f}{2} P_1 (V_2 - V_1) - P_1 (V_2 - V_1) = -\left(\frac{f + 2}{2}\right) P_1 (V_2 - V_1)$$

(Notice the sign changes; writing expressions this way will help in the next part when we sum over the entire cycle.).

For the entire cycle:

To find the values of W, Q and ΔU for the entire cycle, sum the contributions from the four steps :

$$W_{total} = 0 - P_2 (V_2 - V_1) + 0 + P_1 (V_2 - V_1) = (P_1 - P_2)(V_2 - V_1) = - (P_2 - P_1)(V_2 - V_1)$$

$$\Delta U_{total} = \frac{f}{2} V_1 (P_2 - P_1) + \frac{f}{2} P_2 (V_2 - V_1) - \frac{f}{2} V_2 (P_2 - P_1) - \frac{f}{2} P_1 (V_2 - V_1) =$$

$$\frac{f}{2} (P_2 - P_1) (V_1 - V_2 + V_2 - V_1) = 0$$

$$Q_{total} \; = \; - \, W \; = \; (P_2 - P_1)\,(V_2 - V_1)$$

Do these results make sense? First, we expect that ΔU for the entire cycle should be zero since we begin an end at the same values of P and V; by the ideal gas law, T is determined by these values, so we should have the same value of T at the start and end of the cycle. Since U depends on T, no change in T means there is no change in U.

Notice that the work is just the area of the rectangle in the PV diagram. Since the expansion (negative work) occurred at a higher pressure than the compression (the positive work), the total work is negative. Since $\Delta U = 0$, Q = - W and we reproduce this as we expect.

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4. Text, problem 1.35

Solution : We start with eq. (1.39) :

$$V \, T^{f/2} \; = \; cst$$

where cst means a constant. Now, since we want to eliminate T in favor of P, we use the ideal gas law :

$$T = P \, V/N \, k \; = cst^* \, PV.$$

These steps are correct since 1/Nk is just another constant, and a constant times a constant is just another constant.(We will just call all of these cst since we are interested in the functional relationship, and not the value of the constant).Thus :

$$V \, (P \, V \, * \, cst)^{f/2} \; = \; cst \; \Rightarrow \; P^{f/2} \, V^{\left(\frac{2+f}{2}\right)} = cst$$

Exponentiate both sides of the equation to the power of 2/f :

$$\left(P^{f/2} \, V^{\left(\frac{2+f}{2}\right)} \right)^{2/f} = cst^{2/f} \; \Rightarrow \; P \, V^{\left(\frac{2+f}{f}\right)} = \; cst \; (\text{a constant raised to a power is still a constant})$$

and setting $(f + 2)/f \equiv \gamma$, we have :

$$PV^{\gamma} \; = \; cst$$

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5. Text, problem 1.36, parts a) and c). Five pts per part.

Solution : This is a very typical problem dealing with adiabatic compression. The information about temperature (room temp) and that air is a mixture of diatomic molecules allows us to conclude that the number of degrees of freedom for this system will be five (since diatomic molecules have 3 translational modes, 2 rotational and the vibrational modes are frozen out at room temps). Using the equations for adiabatic process we obtain:

a) PV^γ = constant or $P_1 V_1{}^\gamma = P_2 v_2{}^\gamma$

where the subscripts 1 and 2 refer, respectively, to the initial and final states. We can rewrite this equation as :

$$\left(\frac{P_2}{P_1}\right) = \left(\frac{V_1}{V_2}\right)^\gamma$$

Since f = 5, γ = (5 + 2) 5 = 7/5. We are told that the final pressure is 7 times atmospheric pressure, so the ratio on the left is 7 and we have :

$$7 = \left(\frac{V_1}{V_2}\right)^{7/5} \Rightarrow \frac{V_1}{V_2} = 7^{5/7} = 4.01 \Rightarrow \frac{V_2}{v_1} = 0.25$$

or that the final volume is 0.25 L (since the initial volume was 1 L).

c) To find the final temperature, we make use of the equation :

$$V_2 T_2{}^{f/2} = V_1 T_1{}^{f/2}$$

Substituting values :

$$(0.25 \text{ L}) T_2{}^{5/2} = (1 \text{ L}) (300 \text{ k})^{5/2} \text{ or } T_2 = 522 \text{ K}$$

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6. Text, problem 1.40 both parts. Ten points per part.

Solutions:

a) In this problem, we will make use of the basic relations we have learned about thermodynamics, equipartition, and ideal gases. In adiabatic expansion, Q = 0 so we can write :

$$\Delta U = W$$

In this case, the work is due to expansion, so we have :

$$dU = - P dV$$

Appealing to the equipartition theorem, we write :

$$U = \frac{f}{2} N k T = \frac{f}{2} PV$$

Taking differentials :

$$dU = \frac{f}{2} N k dT = \frac{f}{2} (P dV + V dP) \Rightarrow N k dT = P dV + V dP$$

But, since we know that dU = - P dV :

$$N k dT = \frac{- f}{2} N k dT + V dP$$

where we have set - P dV = (f/2) N k dT. Collecting terms gives us :

$$N\,k\,dT\left(1+\frac{f}{2}\right)\;=\;V\,dP$$

Using the ideal gas law to eliminate V from the right hand side :

$$N\,k\,dT\left(1+\frac{f}{2}\right)\;=\;\frac{N\,k\,T\,dP}{P}$$

Canceling terms and rearranging :

$$\frac{dT}{dP}\;=\;\frac{2}{2+f}\,\frac{T}{P}$$

b) We begin by employing the chain rule :

$$\frac{dT}{dz}\;=\;\frac{dT}{dP}\cdot\frac{dP}{dz}$$

We know dT/dP from part a), and dP/dz from our past homework set in which we showed that :

$$P(z)\;=\;P_0\,e^{-\mu\,m_H\,g\,z/(k\,T)}$$

Taking the derivative of this expression for P (z) gives us :

$$\frac{dP}{dz}\;=\;-\frac{\mu\,m_H\,g}{k\,T}\,P$$

Using this expression for dP/dz and our answer in a) for dT/dP, and we find our expression for the **dry adiabatic lapse rate**:

$$\frac{dT}{dz}\;=\;\frac{dT}{dP}\cdot\frac{dP}{dz}\;=\left(\frac{2}{2+f}\,\frac{T}{P}\right)\left(-\frac{\mu\,m_H\,g}{k\,T}\,P\right)\;=\;\frac{-2}{2+f}\left(\frac{\mu\,m_H\,g}{k}\right)$$

Our expression for dT/dz involves only constants with the values :

f = 5 for a diatomic gas a room temp

$\mu\;=\;29$

$m_H = 1.67\times10^{-27}$ kg

$g = 9.8\,\text{m s}^{-2}$

$k\;=\;1.38\times10^{-23}\,\text{J / K}$

We can calculate the value of the lapse rate :

$$\frac{dT}{dz}\;=\;\frac{-2}{7}\left(29\times1.67\times10^{-27}*9.8/1.38\times10^{-23}\right)^{\circ}\text{K / m}\;=\;-0.0098\,^{\circ}\text{K / m}\;\approx\;-10^{0}\,\text{K / km}$$

This means that a rising parcel of air will cool at the rate of 10 C / km as long as the parcel does

not condense (form clouds). We consider the case of condensation in question 7 below.

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7. In question 6, you calculated the lapse rate (dT/dz) for unsaturated air (meaning air that has not condensed). When water vapor condenses into the liquid phase, 2260 J of energy are released for each gram of water vapor that condenses. Predict whether the lapse rate for a parcel of air that has saturated (formed clouds) will have a greater, lesser or equal magnitude than the lapse rate you calculated for dry (unsaturated air). Explain your prediction.

Solution : As each gram of vapor condenses into liquid, 2260J of energy are released into the rising parcel of air. This latent heating represents a source of positive Q for the parcel which partially offsets the loss of energy due to expansion work. Thus, the internal energy of the parcel is greater than if no condensation occurred, and the magnitude of the lapse is decreased. In practical terms, saturated parcels have lapse rates in the range of 5 - 8 C/km, depending on how much vapor is in the parcel.