PHYS 328
HOMEWORK #6

Solutions

1. 2.24, p.67 all parts.In part d), do not explicitly calculate the probability by calculating directly the number of accessible microstates; rather, use the results of part c) to estimate the "reasonableness" of obtaining in the neighborhood of 501, 000 heads or 510, 000 heads. 10 points each part.

Solutions :

a) The maximum multiplicity will occur when N + = N - = N/2, so the maximum multiplicity will be :

$$\Omega_{max} = \frac{N!}{N+!\ N-!} = \frac{N!}{(N/2)!\,(N/2)!}$$

Applying Stirling' s approximation, we obtain :

$$\Omega_{max} = \frac{N^N e^{-N}\sqrt{2\pi N}}{\left((N/2)^{N/2} e^{-N/2}\sqrt{2\pi(N/2)}\right)^2} = 2^N\sqrt{\frac{2}{\pi N}}$$

b) We know that the multiplicity peaks in the neighborhood of N/2, so in this region we can set :

$$N+ = (N/2) + x$$
$$N- = (N/2) - x$$

and write the multiplicity as :

$$\Omega = \frac{N!}{(N/2+x)!\,(N/2-x)!} \approx \frac{N^N e^{-N}\sqrt{2\pi N}}{\left(\frac{N}{2}+x\right)^{\frac{N}{2}+x} e^{\frac{-N}{2}+x}\sqrt{2\pi\left(\frac{N}{2}+x\right)}\ \left(\frac{N}{2}-x\right)^{\frac{N}{2}-x} e^{-\left(\frac{N}{2}-x\right)}\sqrt{2\pi\left(\frac{N}{2}-x\right)}}$$

Notice that the exponential terms in the numerator and denominator cancel to 1, leaving :

$$\frac{N^N\sqrt{2\pi N}}{\left(\frac{N}{2}+x\right)^{\frac{N}{2}+x}\sqrt{2\pi\left(\frac{N}{2}+x\right)}\ \left(\frac{N}{2}-x\right)^{\frac{N}{2}-x}\sqrt{2\pi\left(\frac{N}{2}-x\right)}}$$

We can rewrite the terms in the denominator to give us :

$$\frac{N^N \sqrt{2\pi N}}{\left(\frac{N}{2}+x\right)^{\frac{N}{2}} \left(\frac{N}{2}-x\right)^{\frac{N}{2}} \left(\frac{N}{2}+x\right)^{x} \left(\frac{N}{2}-x\right)^{-x} \sqrt{(2\pi)^2 \left(\left(\frac{N}{2}\right)^2 - x^2\right)}} =$$

$$\frac{N^N \sqrt{\frac{N}{2\pi}}}{\left[\left(\frac{N}{2}\right)^2 - x^2\right]^{\frac{N}{2}} \left(\frac{N}{2}+x\right)^{x} \left(\frac{N}{2}-x\right)^{-x} \sqrt{\left(\left(\frac{N}{2}\right)^2 - x^2\right)}}$$

Now, let' s work with the log of this expression for multiplicity :

$$\ln \Omega =$$

$$N \ln N + \ln \sqrt{\frac{N}{2\pi}} - \frac{N}{2} \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] - x \ln\left(\frac{N}{2}+x\right) - (-x) \ln\left(\frac{N}{2}-x\right) - \frac{1}{2} \ln\left(\left(\frac{N}{2}\right)^2 - x^2\right)$$

We will now focus on some of the expressions involving ln and use the fact that x << N to rewrite the expressions :

$$\ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] = \ln\left[\left(\frac{N}{2}\right)^2 \left(1 - \left(\frac{2x}{N}\right)^2\right)\right] \approx 2 \ln\left(\frac{N}{2}\right) - \left(\frac{2x}{N}\right)^2$$

where I have made use of the approximation ln (1 + x) ≈ x for | x | << 1. Using the same approximation allows us to write:

$$\ln\left(\frac{N}{2} \pm x\right) = \ln\left[\left(\frac{N}{2}\right)\left(1 \pm \frac{2x}{N}\right)\right] \approx \ln\left(\frac{N}{2}\right) \pm \frac{2x}{N}$$

Substituting these expressions into our equation for ln Ω gives :

$$\ln \Omega = N \ln N + \ln \sqrt{\frac{N}{2\pi}} - \frac{N}{2}\left(2 \ln\left(\frac{N}{2}\right) - \left(\frac{2x}{N}\right)^2\right) -$$

$$x \ln\left(\frac{N}{2}\right) - \frac{2x^2}{N} + x \ln\left(\frac{N}{2}\right) - \frac{2x^2}{N} - \frac{1}{2}\left(2 \ln\left(\frac{N}{2}\right) - \left(\frac{2x}{N}\right)^2\right) =$$

$$N \ln N + \ln\sqrt{\frac{N}{2\pi}} - N \ln\left(\frac{N}{2}\right) + \frac{2x^2}{N} - x \ln\left(\frac{N}{2}\right) - \frac{2x^2}{N} + x \ln\left(\frac{N}{2}\right) - \frac{2x^2}{N} - \ln\left(\frac{N}{2}\right) + \frac{2x^2}{N^2} \Rightarrow$$

$$\ln\Omega = N \ln 2 - \frac{2x^2}{N} + \ln\sqrt{\frac{2}{\pi N}} + \frac{2x^2}{N^2} \tag{1}$$

The color coding in the penultimate expression shows which terms I combined to obtain the final expression for ln Ω. Since x << N, we can ignore the final term in eq. (1); exponentiating both sides we obtain finally :

$$\Omega = e^{N \ln 2 - 2x^2/N + \ln\sqrt{2/\pi N}} = 2^N \sqrt{\frac{2}{\pi N}}\, e^{-2x^2/N} = \Omega_{max}\, e^{-2x^2/N} \tag{2}$$

This is a Gaussian whose peak is at x = 0 and we can easily see that when x = 0, our expression for the maximum multiplicity agrees with our answer in part a).

c) The e - folding value for this Gaussian occurs when the argument of the exponential equals minus one, or when :

$$-1 = -\frac{2x^2}{N} \Rightarrow x = \sqrt{N/2}$$

d) The result in part c) tells us that for one million flips of a coin, the e - folding height of the multiplicity is :

$$x = \sqrt{500,000} = 707$$

Let' s think of this as close enough to 700. Obtaining 501, 000 heads means that x = 1000, which is a bit more than the “scale height”, and the multiplicity of this macrostate is smaller than the maximum multiplicity by a factor of :

$$e^{-2(1000)^2/10^6} = e^{-2} = 0.13$$

so it seems reasonable that with enough attempts, we could get 501, 000 heads out of 1, 000, 000 flips. However, if the number of heads is 510, 000, our value of x = 10, 000, and the multiplicity is smaller than the maximum value by a factor of:

$$e^{-2(10,000)^2/10^6} = e^{-200} \approx 1.4 \times 10^{-87}$$

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2. 2.25 p.67 parts a) and b) only.The problem is predicated on a random walk model which is also known as the "drunken sailor" problem.A drunken sailor walks out of a bar (sounds like the start of a bad joke*) and is so intoxicated he cannot remember whether his last step was to the left or the right.Thus, each step has a 50/50 chance of being to the left or to the right.Assume the sailor only walks in a straight line and that each step is 1 unit of distance (so we have a constrained drunken sailor problem).Explain your answers and/or show work.10 pts each part.

Solutions : a) Since each step has the same probability of going to the left or right, the most probable ending point is at x=0, or the starting point.

b) In the previous problem, we determined the expression for the Gaussian describing multiplicities near the peak of the distribution. This problem is essentially the same as the coin tossing problem in that there are only two possible events for each trial, and the likelihood of obtaining either outcome is 50/50. Therefore, we can use the Gaussian we had in the previous problem, namely:

$$e^{-2x^2/N} \tag{3}$$

where we should remember, x represents the separation from the peak of the distribution. In this case, this represents not a physical distance, but how many steps greater than N/2 the sailor took in a particular direction. Thus, we can write x in this situation as (assuming the excess is to the right) :

$$x = \left(N_{right} - N/2\right)$$

where N_{right} and N_{left} will represent the total number of steps to the right and to the left respectively.

If the length of each step is L, then the physical displacement from the origin is :

$$\text{displacement} \equiv \Delta = \left(N_{right} - N_{left}\right) L = \left(N_{right} - (N - N_{left})\right) L = \left(2\,N_{right} - N\right) L$$

where I have used the fact that $N = N_{right} + N_{left}$. Now, a little rearrangement of the final expression immediately above gives:

$$\text{displacement} = 2\left(N_{right} - \frac{N}{2}\right) L = 2\,x\,L \Rightarrow x = \Delta / 2\,L$$

In terms of physical displacement from the peak of the Gaussian, eq. (3) becomes :

$$e^{-2(\Delta/2L)^2/N} = e^{-\Delta^2/2L^2 N}$$

The e - folding value for this Gaussian occurs when :

$$1 = \Delta^2 \big/ 2\,L^2\,N \Rightarrow \Delta = L\sqrt{2\,N}$$

For N ≈ 10,000 (a total scalar distance of probably a few miles), the probability falls to 1/e of its maximum value at a distance of $\sqrt{20\,000}$ L or approximately 141 steps. Thus, I would expect the sailor to wind up no more than a few hundred paces from the starting point. The probability of being beyond 140 steps from the origin is approx 1/e, and becomes smaller for greater displacements.

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3. 2.28 p. 77

Solution : The total number of arrangements of 52 different objects is 52! The entropy is then given by:

$$S = k \ln \Omega = k \ln 52! = 156\,k.$$

In natural units, S/k = 156. Multiply this by Boltzmann' s constant to obtain :

$$2.2 \times 10^{-21}\ J/K.$$

This is the amount of entropy generated by shuffling the cards ; the total amount of entropy due to thermal motions will be on the order of N k, so that the shuffling entropy is insignificant compared to the entropy due to thermal motion.

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4. 2.29 p. 77

Solution : This problem will involve repeated use of the definition of entropy :

$$S = k \ln \Omega \text{ or } S/k = \ln \Omega$$

For the most probable state (with 60 units of energy in solid A), there are 6.9 10^{114} total states, so

$$S/k = \ln\left(6.9 \times 10^{114}\right) = 264.4$$

For the least likely state (with all the energy in B) :

$$S/k = \ln\left(2.8 \times 10^{81}\right) = 187.5$$

If all the microstates are allowable, we have :

$$S/k = \ln\left(9.3 \times 10^{115}\right) = 267$$

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5. 2.31 p. 79

Solution : We begin by writing the Sackur-Tetrode equation:

$$\Omega = \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{(3N/2)!} \left(\sqrt{2 m U}\right)^{3N} \tag{4}$$

We know that entropy is proportional to ln Ω, so we will be taking the ln of eq. (4). Since we are dealing with a very, very large number of particles, we can use Stirling' s approximation in the form :

$$\ln N! = N \ln N - N$$

Since we know the final form we wish to reproduce, notice how I try to write ln Ω in terms of N. Taking the ln of both sides :

$$\ln \Omega = N \ln V + N \ln \pi^{3/2} + N \ln (2 m U)^{3/2} -$$
$$(N \ln N - N) - N \log h^3 - ((3N/2) \ln (3N/2) - 3N/2) =$$
$$N\left[\ln V + \ln\left(\frac{2 m U \pi}{h^2}\right)^{3/2} - \ln N + 1 - \ln (3N/2)^{3/2} + 3/2\right]$$

In the step above, I have incorporated the h^3 term inside the $(\quad)^{3/2}$ term since

$$h^3 = \left(h^2\right)^{3/2}$$

Combining lns, we get :

$$S = k \ln \Omega = N k\left[\ln\left[\left(\frac{V}{N}\right)\left(\frac{2 \pi m U}{(3N/2) h^2}\right)^{3/2} + 5/2\right]\right] = N k\left[\ln\left[\left(\frac{V}{N}\right)\left(\frac{4 \pi m U}{3 N h^2}\right)^{3/2} + 5/2\right]\right]$$

Multiply this by k and this is the desired form of the Sackur - Tetrode equation for entropy.

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6. 2.33 p. 79

Solution : This is pretty much a grind and find using the form of the Sackur-Tetrode equation we just derived. For this problem, our parameters are:

$N = 1\,\text{mole} = 6.02 \times 10^{23}$ molecules

$V = N k T / P = 0.025\,m^3$

$m = 40 \times 1.6 \times 10^{-27}$ kg

$U = \frac{1}{2} f N k T = 3726\,J$

$h = 6.63 \times 10^{-34}\,J/s$

$k = 1.38 \times 10^{-23}\,J/K$

Let's write a short *Mathematica* program :

In[137]:=
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n=6 10^23;v = 0.025;m=40 1.6 10^-27;u = 3726;h=6.63 10^-34;k=1.38 10^-23;
factor=(v/n)(4 π m u/(3 n h^2))^(3/2);
entropy = n k(Log[factor]+5/2);
Print["The entropy for one mole of argon gas = ", entropy,"  J/K"]
```

During evaluation of In[137]:=

The entropy for one mole of argon gas = 153.915 J/K

This is larger than the calculation done in the text for a mole of He gas. The principal difference between the two gases is the mass of an atom (Argon is 10 times more massive than He), and this causes each argon atom to have more momentum than a He atom at the same temperature.