# PHYS 301
# HOMEWORK #7

## Solutions

1. Starting with the text' s equations 3.28 and 3.29, complete the algebraic steps to derive equations 3.30 and 3.31

Solution : We begin with :

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial N+}{\partial U}\frac{\partial S}{\partial N+} = \frac{-1}{2\,\mu\,B}\frac{\partial S}{\partial N+}$$

where we made use of the definition U = $\mu$B (N - 2 N +). (Please note that I am using "+" rather than the up arrow for ease of typing). Now, eq. 3.28 expresses S in terms of N + :

$$\frac{S}{k} = N \ln N - N + \ln N + - (N - N+) \ln (N - N+)$$

Taking the derivative we get :

$$\frac{1}{T} = \frac{-1}{2\,\mu\,B} \cdot k \frac{\partial}{\partial N+}[N \ln N - N + \ln N + - (N - N+) \ln (N - N+)]$$

$$\frac{1}{T} = \frac{-k}{2\,\mu\,B}\left[-\ln N + - \frac{N+}{N+} - \frac{(-1)(N - N+)}{(N - N+)} + \ln (N - N+)\right]$$

$$\frac{1}{T} = \frac{-k}{2\,\mu\,B}[-\ln N + -1 + 1 + \ln (N - N+)] = \frac{-k}{2\,\mu\,B} \ln \frac{N - N+}{N+} = \frac{+k}{2\,\mu\,B} \ln\left(\frac{N+}{N - N+}\right) \qquad (1)$$

We can rewrite the text' s eq. 3.25 as :

$$N+ = \frac{N}{2} - \frac{U}{2\,\mu\,B}$$

Substituting this into our eq. 1, gives us :

$$\frac{1}{T} = \frac{k}{2\,\mu\,B} \ln\left(\frac{N - U/\mu\,B}{N + U/\mu\,B}\right)$$

and this is eq. 3.30. Now we solve for U to derive eq. 3.31 :

$$\ln\left(\frac{N - U/\mu\,B}{N + U/\mu\,B}\right) = \frac{2\,\mu\,B}{k\,T} \equiv Z$$

I define this parameter Z for ease of typing. Exponentiating, we get :

$$\frac{N - U/\mu B}{N + U/\mu B} = e^Z$$

Multiplying through and collecting U terms gives :

$$\frac{U}{\mu B}\left(e^Z + 1\right) = N\left(1 - e^Z\right) \Rightarrow U = \mu B N \frac{\left(1 - e^Z\right)}{\left(1 + e^Z\right)}$$

which is the intermediate expression in eq. 3.31. To turn this into hyperbolic functions, multiply numerator and denominator by Exp[-Z/2], so we get :

$$U = \mu B N\left(e^{-Z/2} - e^{Z/2}\right) \Big/ \left(e^{-Z/2} + e^{Z/2}\right)$$

We can see that the numerator is - 2 sinh (Z/2) and the denominator is 2 cosh (Z/2), so our expression becomes :

$$U = -\mu B N \sinh(Z/2)/\cosh(Z/2) = -\mu B N \tanh(Z/2) = -\mu B N \tanh(\mu B/k T).$$

Which is the final expression desired.

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2. In the limit that $\mu$B/k T << 1, use appropriate series expansions to show that eq. 3.35 follows from eq. 3.32.

Solution : In the high temperature limit, the argument of the tanh function becomes small, and we can write expand the exponentials as :

$$M = N\mu \tanh(\mu B/k T) = N\mu \frac{\left(e^X - e^{-X}\right)}{\left(e^X + e^{-X}\right)} \quad \text{where } X = \mu B/k T$$

When X << 1, we can expand :

$$M = N\mu[(1 + X - (1 - X))/(1 + X + (1 - X))] = \frac{2 N \mu X}{2} = N\mu X = N\mu^2 \Big/ k T$$

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3. Problem 3.25 parts a) - d). 10 pts for b), 5 pts for each of a), c), d).

***Solutions*** : a) We know that S = k ln Ω, so we have :

$$S = k \ln\left[\left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N\right] = k q \ln\left(\frac{q+N}{q}\right) + k N \ln\left(\frac{q+N}{N}\right)$$

b) We find the temperature as a function of U using :

$$\frac{1}{T} = \frac{\partial S}{\partial U}$$

We can simplify this a bit knowing that U = q $\epsilon$ and using the chain rule :

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial S}{\partial q} \cdot \frac{\partial q}{\partial U} = \frac{1}{\epsilon}\frac{\partial S}{\partial q}$$

the last step resulting from the fact that dU/dq = $\epsilon$. We now use the expression in part a) and take the derivative of S with respect to q :

$$\frac{1}{T} = \frac{1}{\epsilon}\frac{\partial S}{\partial q} = \frac{k}{\epsilon}\frac{\partial}{\partial q}[\, q \ln (q+N) - q \ln q + N \ln (q+N) - N \ln N]$$

$$\frac{1}{T} = \frac{k}{\epsilon}\left[\ln (q+N) + \frac{q}{q+N} - \ln q - 1 + \frac{N}{q+N}\right]$$

$$\frac{1}{T} = \frac{k}{\epsilon}\left[\ln\left(\frac{q+N}{q}\right) + \left(\frac{q+N}{q+N}\right) - 1\right] = \frac{k}{\epsilon}\ln\left(\frac{q+N}{q}\right)$$

Solving in terms of T gives us :

$$T = \frac{\epsilon}{k \ln (1 + N/q)} = \frac{\epsilon}{k \ln (1 + N\epsilon/U)}$$

Exponentiate both sides and solve for U :

$$U = \frac{N\epsilon}{e^{\epsilon/kT} - 1}$$

c) To find the heat capacity, we find dU/dT :

$$C = \frac{d}{dT}\left(\frac{N\epsilon}{e^{\epsilon/kT} - 1}\right) = \frac{-1\, N\epsilon}{\left(e^{\epsilon/kT} - 1\right)^2}\frac{d}{dT}\left(e^{\epsilon/kT}\right) = \frac{-1\, N\epsilon\, e^{\epsilon/kT}}{\left(e^{\epsilon/kT} - 1\right)^2}\left(\frac{-\epsilon}{kT^2}\right) = \frac{N\epsilon^2}{kT^2}\frac{e^{\epsilon/kT}}{\left(e^{\epsilon/kT} - 1\right)^2}$$

d) When T is large, the argument of the exponential is small, and we can expand the exponent in the denominator in a Taylor series keeping only the first order terms :

$$C = \frac{N\epsilon^2}{kT^2}\left(\frac{1}{1 + (\epsilon/kT) - 1}\right)^2 = \frac{N\epsilon^2}{kT^2}\cdot\frac{1}{(\epsilon/kT)^2} = Nk$$

As T → ∞, we expect all the possible degrees of freedom to be unfrozen; recalling that this derivation was done for an Einstein oscillator, and that each Einstein oscillator has 2 degrees of freedom, this is exactly the result we expect from the equipartition theorem which predicts that U = (f/2) N k T. For f = 2, we have that C = dU/dT = N k.

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4. Problem 3.32 (all parts; each part 5 pts).

Solutions : a) This is the simplest of all work problems : Work = force x displacement.

$$W = 2000\,N \cdot 10^{-3}\,m = 2\,J$$

b) Remember that heat is the flow of energy between two objects of different temperatures in context. This situation does not exist, so there is no heat flow in this scenario.

c) Thus, the internal energy of the gas increases by the work done, or by 2 J. Since the work is done on the gas, the internal energy increases (and the temperature would increase).

d) The thermodynamic identity gives us :

$$T\,ds = dU + P\,dV \Rightarrow dS = \frac{1}{T}\,dU + \frac{P}{T}\,dV$$

We know $T = 300\,K$, $dU = 2\,J$, $P = 10^5\,N/m^2$;

so we have to find dV. We are given the distance the piston moves and the area of the piston, so we can compute easily :

$$dV = -10^{-3}\,m \cdot 0.01\,m^2 = -10^{-5}\,m^3.$$

Since the volume is decreasing, be sure to use the proper sign. Substituting all relevant values, we get :

$$dS = \frac{1}{300\,K}\left(2\,J + 10^{-5}\,N/m^2\left(-10^{-5}\,m^3\right)\right) = \frac{1}{300}\,J/K = 0.0033\,J/K$$

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5. Problem 3.36 both parts. 5 pts a), 10 pts b).

a) We use the expression for $\Omega$ from problem 3 and compute the entropy:

$$S = k\ln\Omega = k\ln\left[\left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N\right] = k\,q\ln\left(\frac{q+N}{q}\right) + k\,N\ln\left(\frac{q+N}{N}\right)$$

To find the chemical potential, we take :

$$\mu = -T\left(\frac{\partial S}{\partial N}\right) = -T\left[k\frac{\partial}{\partial N}\left(q\ln\left(\frac{q+N}{q}\right) + k\,N\ln\left(\frac{q+N}{N}\right)\right)\right]$$

$$\mu = -k\,T\left[\frac{q}{q+N} + \frac{N}{q+N} + \ln(q+N) - \ln N - \frac{N}{N}\right] = -k\,T\left[\ln\left(\frac{q+N}{N}\right)\right]$$

b) If N >> q, we can write the ln that appears in the expression for chemical potential as :

$$\ln\left(\frac{q+N}{N}\right) = \ln\left(1+\frac{q}{N}\right) \approx \frac{q}{N} \text{ using the approximation that } \ln(1+x) \approx \text{ for x small.}$$

In the limit where N >> q, we see that the increase in entropy goes roughly as q/N, a very small number. Does this make sense? Remember that entropy is a measure of the number of accessible states to a system. The change in entropy is thus the change in number of states, and the chemical potential is a measure of how the number of states changes as we add particles. So, consider a very simple system consisting of 1000 particles sharing 1 unit of energy. We know that there are only 1000 ways this 1 unit of energy can be distributed among the 1000 particles. If we add 1 more particle then the number of accessible states is 1001, and the entropy has increased slightly, and we can approximate $\partial S/\partial N \sim 1/1000$, or, q/N.  Thus the chemical potential should vary roughly as - k T q/N.  So this result makes sense (although I would not say it is glaringly intuitive).

In the limit where $q \gg N$, the ln in the chemical potential expression approaches the value of ln (q/N).  Now, suppose we have a system that starts out with 1000 units of energy and only 1 particle. There is only 1 way to distribute this energy among the particles, i.e., all the energy is in the 1 particle.  But now, if we add a particle, we have 1001 ways of distributing energy between the 2 particles (0 energy in particle A, 1 unit of energy in particle A, 2 units ....usw, usw, usw). Recall that chemical potential tells you how the entropy changes as the number of particles changes, and we can see that $\partial S/\partial N$ is quite large for our hypothetical system of 1000 units of energy and 1 particle; if we add just one more particle, the magnitude of $\partial S/\partial N$ is large.  Thus, the chemical potential of the system with q=1000 and N=2 is much more negative than the system where q=1000 and N =1. Since there are many more ways to distribute energy to 2 particles than to 1, the (slightly) denser system has a greater entropy, and therefore a greater probability to occur.  Again, this makes sense, but I would not call it intuitively obvious.