PHYS 328
HOMEWORK #8

Due : 1 Nov. 2012

1. Consider a heat engine cycle consisting of :

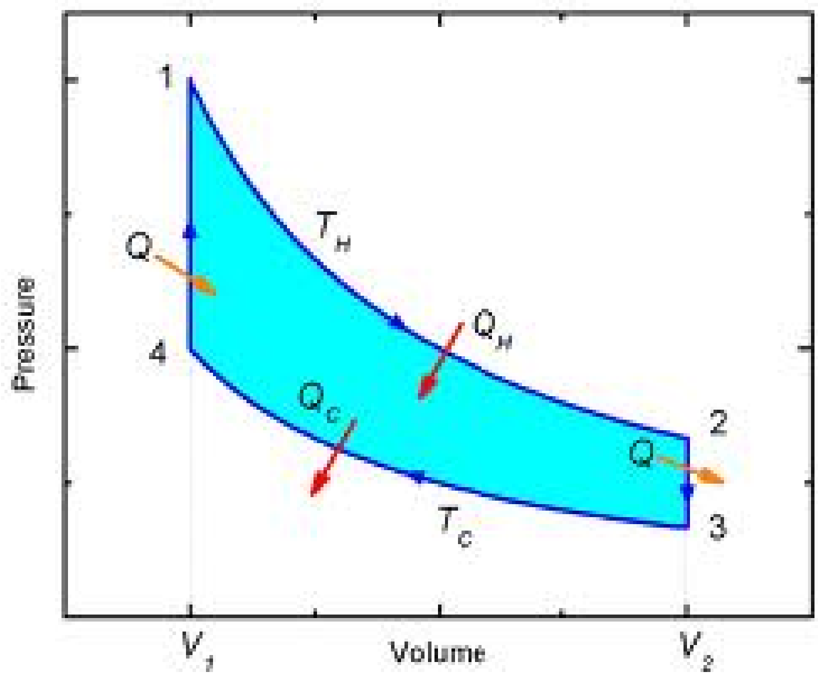
Step 1 : Isothermal expansion at temperature T_h.
Step 2: Removing heat at constant volume until the temperature reaches T_C.
Step 3: Isothermal compression at T_C.
Step 4: Heating the gas at constant volume until the temperature returns to T_h and the cycle renews.
(The PV diagram for this process should be equivalent to the process described in question 1 of the first hour exam).

Find the efficiency of this heat engine, and compare its efficiency explicitly to the efficiency of a Carnot engine operating between these two temperatures.Show explicitly that the efficiency of this engine is less than a Carnot engine's.

Solution : We know that the efficiency of a heat engine can be expressed as :

$$e = \frac{W}{Q_h}$$

where W is the total work done in one cycle and Q_h is the heat extracted from the hot reservoir. The diagram below (painstakingly transported from the internet) will serve as our benchmark for analyzing this problem.



The power stroke, corresponding to the isothermal expansion of the gas occurs between points 1 and 2 in the diagram above; heat is extracted from the gas between points 2 and 3 until the temperature of the gas drops until it is infinitesimally above the temperature of the cold reservoir. The gas is isothermally compressed between points 3 and 4, and is then heated at constant volume until its temperature is just less than the temperature of the hot reservoir. The area of the curve represents the total work done by the gas, and the total heat extracted from the hot reservoir is the sum of the Q values for steps $1 \to 2$ and steps $4 \to 1$.

There is no work done in the constant volume steps, thus the total work is :

$$W = W_{12} + W_{34} = N k T_h \int_{V_1}^{V_2} \frac{dV}{V} + N k T_c \int_{V_2}^{V_1} \frac{dV}{V} = N k (T_h - T_c) \ln \frac{V_2}{V_1}$$

The total heat extracted is :

$$Q_h = Q_{12} + Q_{41}$$

In the isothermal expansion, we know that the change in internal energy is zero (since the temp does not change); thus, the first law of thermo shows that the heat extracted equals the work done :

$$Q_{12} = W_{12} = N k T_h \ln \frac{V_2}{V_1}$$

Recall that our definitions of Q and W used in the efficiency equations mean we care only about positive quantities. In the constant volume heating, there is no work done, so the heat extracted equals the change in internal energy of the process, or :

$$Q_{41} = U_1 - U_4 = \frac{f}{2} N k (T_h - T_c)$$

We can now compute the efficiency of this type of engine as :

$$e = \frac{W}{Q_h} = \frac{N k (T_h - T_c) \ln \frac{V_2}{V_1}}{N k T_h \ln \frac{V_2}{V_1} + \frac{f}{2} N k (T_h - T_c)}$$

The second part of the problem asks you to show that this is less than the efficiency of the Carnot cycle, and a number of you indicated some difficulty in algebraically equating the two expressions for efficiency. Suppose we consider the reciprocals of efficiency; we would write the inverse of efficiency for this engine as :

$$\frac{1}{e} = \frac{N k T_h \ln \frac{V_2}{V_1} + \frac{f}{2} N k (T_h - T_c)}{N k (T_h - T_c) \ln \frac{V_2}{V_1}} = \frac{T_h}{T_h - T_c} + \frac{f}{2 \ln \frac{V_2}{V_1}}$$

The first term on the right of the final expression is just 1/(efficiency of a Carnot engine), so we have :

$$\frac{1}{e} = \frac{1}{e_{Carnot}} + \frac{f}{2 \ln \frac{V_2}{V_1}}$$

Since the second term is always positive, we have that 1/e < 1/e (Carnot), which implies that e (Carnot) > e (this engine). **QED**. This type of engine is known as a *Stirling Engine*.

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2. Why are air conditioning units placed in windows and not the middle of a room?

Solution: An air conditioner has to dump its excess heat (and entropy) somewhere. If it is dumped inside the room or building, the waste heat will warm the room instead of cooling it. Thus, air conditioners are mounted in windows to vent excess heat to the outside environment.

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3. Problem 4.1.

Solution : We start again with the definition of efficiency :

$$e = \frac{W}{Q_h}$$

The total work done is just the area of the rectangle, which will be :

$$W = (P_2 - P_1)(V_2 - V_1) = 2 P_1 V_1 \text{ (for the case where } P_2 = 2 P_1 \text{ and } V_2 = 3 V_1)$$

Heat is absorbed in the constant volume pressurization (step A in Fig. 1.10 (b) on p. 23) and in the expansion phase of the cycle (step B). Refer back to the solutions to homework #3, problem 3, to obtain expressions for Q in steps A and B (and recall that f = 5 for a diatomic gas) :

$$Q_h = Q_A + Q_B = \frac{5}{2} V_1 (P_2 - P_1) + \frac{7}{2} P_2 (V_2 - V_1) = \frac{5}{2} P_1 V_1 + \frac{7}{2} (2 P_1 (2 V_1)) = \frac{33}{2} P_1 V_1$$

Therefore, the efficiency of this process is :

$$e = \frac{2 P_1 V_1}{(33/2) P_1 V_1} = \frac{4}{33} = 12\,\%$$

To compute the efficiency of a Carnot cycle, we need to know the high and low temperatures in a cycle. Using the ideal gas law as our guide, the lowest temperature will occur when the product of PV is the smallest; or :

$$T_c = \frac{P_1 V_1}{N k}$$

The highest operating temperature occurs when P and V are simultaneously a maximum, so that :

$$T_h = \frac{P_2 V_2}{N k} = \frac{6 P_1 V_1}{N k}$$

Thus, the efficiency of a Carnot engine operating between these temperatures is :

$$e_{Carnot} = \frac{6 P_1 V_1 - P_1 V_1}{6 P_1 V_1} = \frac{5}{6} = 83.3\,\%$$

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4. Problem 3.31

Solution : We begin with :

$$\Delta S = \int_{T_1}^{T_2} \frac{C_P}{T}\, dT = \int_{T_1}^{T_2} \frac{\left(a + b\,T - c T^{-2}\right)}{T}\, dT = a \ln\left(\frac{T_2}{T_1}\right) + b\,(T_2 - T_1) + \frac{c}{2}\left(\frac{1}{T_2^2} - \frac{1}{T_1^2}\right)$$

Substituting the values given by the problem :

$a = 16.86\,\mathrm{J/k};\ b = 4.77 \times 10^{-3}\,\mathrm{J}/\mathrm{K}^2;\ c = 8.54 \times 10^5\,\mathrm{J \cdot K};\ T_2 = 500\,\mathrm{K};\ T_1 = 298\,\mathrm{K}$

gives us :

In[11]:=
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Clear[a,b,c,t1,t2,deltaS,cp]
a=16.86;b=0.00477;c=8.54 10^5;t1=298;t2=500;
cp[temp_]:=a + b temp -c/temp^2
deltaS=Integrate[cp[temp]/temp,{temp,298,500}];
Print["The change in entropy from 298K to 500K = ",deltaS, " J/K"]
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The change in entropy from 298K to 500K = 6.588494390338951` J/K
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This is how much the entropy changes between these two temps; since the entropy is 5.74 J/K at

298 K (see table at back of book), the entropy at 500 K is simply 6.59 J/K + 5.74 J/K = 12.3 J/K

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5. Problem 5.8

Solution: Start with the thermodynamic identity from the end of Chapter 3 :

$$dU = T\,dS - P\,dV + \mu\,dN \tag{1}$$

I know that U is related to F via :

$$F = U - T\,S \Rightarrow dF = dU - T\,dS - S\,dT \tag{2}$$

and that F and G are related by :

$$G = F + P\,V \Rightarrow dG = dF + P\,dV + V\,dP \tag{3}$$

Using our expression for dF from equation (2) in equation (3) :

$$dG = (dU - T\,dS - S\,dT) + P\,dV + V\,dP \tag{4}$$

Substituting for dU from equation (1) into eq. (4) :

$$\mathbf{dG} = (T\,dS - P\,dV + \mu\,dN - T\,dS - S\,dT) + P\,dV + V\,dP = \boldsymbol{\mu}\,\mathbf{dN - S\,dT + V\,dP} \tag{5}$$

Partial differentiation gives us :

$$\left(\frac{\partial G}{\partial N}\right)_{T,P} = \mu \qquad \left(\frac{\partial G}{\partial T}\right)_{N,P} = -S \qquad \left(\frac{\partial G}{\partial P}\right)_{P,N} = V$$

The first of these partial derivatives gives us an interesting insight into the meaning of μ. If we integrate both sides to obtain G = μ N, we can think of chemical potential, μ, simply as Gibbs' Free Energy/particle.