PHYS 328
HOMEWORK #9--Solutions

1. Inspired by the first problem of the last homework set :

Solve the differential equation (without using Mathematica or other software programs) :

$$\frac{dy}{dx} = \frac{1}{x+y}$$

The inspiration does not derive from the content of the question, but from seeing how a simple algebraic "trick" can turn a seemingly difficult problem into a simple one. What simple "trick" can you do to turn this into an almost trivial ODE?

Solution : The differential equation above looks tough, but if we simply invert each side we get :

$$\frac{dx}{dy} = x + y \Rightarrow \frac{dx}{dy} - x = y$$

This is an elementary ODE solved by simple methods, yielding x (y) as :

$$x(y) = c\,e^{y} - y - 1$$

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2. 3.34, parts a) - e). 5 pts for a), b), c), e); 10 pts for d)

Solutions : a) The problem is essentially the same as flipping coins or a paramagnet, and we can write the number of ways of getting N_R links to the right out of a total of N links as :

$$\Omega = \binom{N}{N_R} = \frac{N!}{N_R!\,(N-N_R)!}$$

The entropy of this system is then :

$$S = k \ln \Omega = k\,[\ln N! - \ln N_R! - \ln (N-N_R)!]$$

Making use (as we have many times) Stirling' s approximation for very large values of N, we can expand the ln terms and obtain (after a little algebra) :

$$S = k[N \ln N - N_R \ln N_R - (N-N_R) \ln (N-N_R)] \tag{1}$$

b) If there are as many links to the left as right, the total length of the chain, L, is zero. Therefore, the total length will be equal to the excess of links to the right (we are assuming a right handed rubber hand) times the average length of a link, therefore we can write L as :

$$L = (N_R - N_L)\,l = (2\,N_R - N)\,l \tag{2}$$

where the last step recognizes that the sum of links to the left and links to the right is N.

c) The thermodynamic identity tells us :

$$dU = T\,dS - P\,dV$$

Following the text's hints, we know that L is analogous to V, P is analogous to $-$ F, so we obtain :

$$dU = T\,dS + F\,dL$$

Notice that this is what we would expect from the first law. The change in energy (dU) equals the heat that enters the system (T dS) and the work done on the system (+F dL).

d) Following a derivation similar to the reasoning that led to eq. 3.39 in the text, we can substitute L for V and $-$ F for P to obtain :

$$F = -T\left(\frac{\partial S}{\partial L}\right)_V \tag{3}$$

Combining equations 1 and 2, we can take the indicated partial derivative and obtain the desired relationship :

$$\frac{\partial S}{\partial L} = \frac{\partial S}{\partial N_R}\frac{\partial N_R}{\partial L} = \frac{1}{2l}\frac{\partial S}{\partial N_R} = \frac{1}{2l}\left[-\ln N_R - 1 + \ln(N - N_R) + \frac{N}{N - N_R} - \frac{N_R}{N - N_R}\right]$$

$$\Rightarrow \frac{\partial S}{\partial L} = \frac{1}{2l}\ln\frac{(N - N_R)}{N_R}$$

Using this expression for $\partial S/\partial L$, we obtain an expression for F :

$$F = -T\left(\frac{\partial S}{\partial L}\right)_V = -\frac{kT}{2l}\ln\frac{N - N_R}{N_R}$$

d) We will need to rewrite equation 4 using equation 2 :

$$L = (2N_R - N)\,l \Rightarrow N_R = \frac{1}{2}\left(\frac{L}{l} + N\right)$$

Substituting this into equation 4 :

$$F = -\frac{kT}{2l}\ln\left(\frac{N}{N_R} - 1\right) = -\frac{kT}{2l}\ln\left(\frac{N}{(L/l + N)/2} - 1\right) = -\frac{kT}{2l}\ln\left(\frac{2}{L/(Nl) + 1} - 1\right) =$$

$$-\frac{kT}{2l}\ln\left(\frac{1 - L/Nl}{1 + L/Nl}\right) = +\frac{kT}{2l}\ln\left(\frac{1 + L/Nl}{1 - L/Nl}\right)$$

In the case where L << N l, we can expand the fraction knowing 1/(1 - x) = 1 + x for small values of x, and then use ln (1 + z) $\approx$ z for small z, giving us :

$$F = \frac{k\,T}{2\,l} \cdot 2\,\frac{L}{l\,N} = \frac{k\,T\,L}{l^2\,N}$$

Notice that the force is proportional to extension (L); this is Hooke' s law (F = cst ·x) where the constant in this case is k $T \big/ \left(l^2 N\right)$

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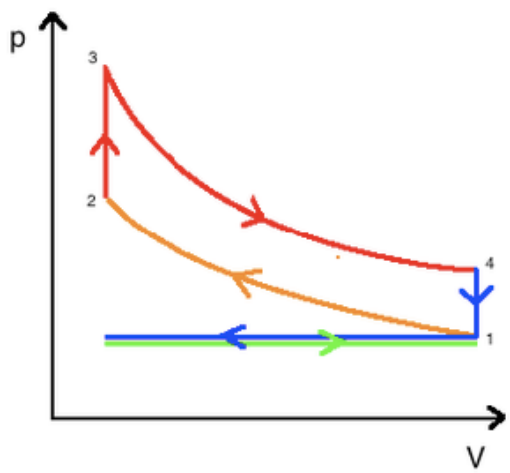
3. Suppose the cycle represented in the first homework of the last problem set consisted of two adiabatic steps separated by constant volume steps (i.e., steps 1 → 2 and 3 → 4 are adiabatic instead of isothermal). Determine an expression for the efficiency of this cycle. You should be able to express your final answer in terms of V and γ. (this is an Otto engine and you can read about it in section 3 of chapter 4).

Solution : Let' s use the diagram below in our discussion :



Since the paths 3 -> 4 and 1 -> 2 are adiabatic, there is no heat extracted (or lost) during those steps. Thus, we know that heat is extracted from the hot reservoir only during step 2 -> 3 and heat is dumped into the cold reservoir during step 4 -> 1. We can write the efficiency of this cycle as :

$$e = 1 - \frac{Q_c}{Q_H}$$

Since now work is done in the constant volume steps, the first law of thermodynamics tells us that :

$$Q_C = \Delta U_{41} = \frac{f}{2}\,N\,k\,(T_h - T_C) = \frac{f}{2}\,N\,k\,V_1\,(P_4 - P_1)$$

$$Q_H = \Delta U_{32} = \frac{f}{2}\,N\,k\,(T_3 - T_2) = \frac{f}{2}\,V_2\,(P_3 - P_2)$$

The required ratio is then :

$$\frac{Q_c}{Q_H} = \frac{V_1\,(P_4 - P_1)}{V_2\,(P_3 - P_2)}$$

But since we know the processes are adiabatic, we know that the gas follows the relationships :

$$P_3 V_2^{\gamma} = P_4 V_1^{\gamma} \text{ and } P_1 V_1^{\gamma} = P_2 V_2{}^{\gamma}$$

Writing the P' s in terms of V and substituting, we get :

$$e = 1 - \frac{Q_c}{Q_H} = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

the ratio $\frac{V_2}{V_1}$ is called the compression ratio of the engine.

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4. 5.10 from the text

Solution : We use the identity :

$$\left(\frac{\partial G}{\partial T}\right)_P = -S$$

to obtain that dG = - S d T. Using the data from the Appendix of the book, we have that S = 70 J/K and dT = 5 K, so that dG = - 350 J. To find the change in Gibbs Free Energy when we change pressure, use :

$$\left(\frac{\partial G}{\partial P}\right)_T = V$$

and solve for d G = V d P ⇒ dp = dG/V. We want to compenstate for the 350 J loss in dG due to heating, so we calculate the volume of 18 g of water (the density of water is 1000 kg per cubic meter, so the volume of 0.018 kg = 1.8 10^{-5} cubic meters, and we have that:

$$dP = 350\,J / 1.8\ 10^{-5}\,m^3 = 1.9 \times 10^7 \text{ bars}$$

. 5.12 from the text; derive only the relationship :

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$$

Solution : Since the partial derivatives found in physics tend to be fairly well behaved, we can usually be confident that the order of differentiating does not matter, thus :

$$\frac{\partial}{\partial V}\left(\frac{\partial U}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial U}{\partial V}\right)$$

The thermodynamic identity tells us :

$$dU = T\,dS - P\,d\,V$$

so that :

$$\frac{\partial U}{\partial S} = T \text{ at constant } V \quad \text{and} \quad \frac{\partial U}{\partial V} = -P \text{ at constant } S.$$

Therefore :

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$$

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6. Problem 6.6 in text.

Solution : Consider an electron in the first excited state (n = 2); the probability of being in any one of the four possible n = 2 states is :

$$P(n = 2) = \frac{e^{-E_2/kT}}{Z}$$

where Z is the partition function. Since the energy difference between the first excited state and ground state is large compared to k T at either room temp or at 9500 K, we can approximate the partition function, Z, by using only the Boltzmann factor for the ground state :

$$P(n = 2) = \frac{e^{-E_2/kT}}{e^{-E_1/kT}} = e^{-(E_2-E_1)/kT}$$

The energy difference between the two states is 10.2 ev = 1.6 10^{-18}J, so we have:

$$P(n = 2) = e^{-118{,}260/T}$$

For T = 300 k, the argument of the exponential is 394 and the probability is then essentially zero for finding an electron in the first excited state of H at room temp. In the atmosphere of UMa (T = 9500 K), the argument of the exponential is 12.4, so the probability of finding an electron in any one of the four n = 2 states is $e^{-12.4}$ = 3.9 10^{-6}. Since there are four available n=2 states, the total probability of finding an electron in n=2 in UMa is 4 times this, or 1.6 10^{-5}.

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7. Problem 6.12 in text.

Solution : As shown in the problem above the probability of being in an excited state compared to the ground state is :

$$P(n > 1 / n = 1) = e^{-(E_u-E_1)/kT}$$

where E_u is the energy of an upper level and E_1 is the energy of the ground state. We are told that this probability is 1/10 and that the energy difference between the ground and rotational states is 4.74 10^{-4} ev (= 7.6 10^{-23}J). Thus, we have that:

$$0.1 = e^{-7.6\,10^{-23}/kT} \Rightarrow 0.1 = e^{-5.5/T}$$

Taking ln of both sides and solving for T :

$$T = \frac{5.5}{\ln 10} = 2.4\,\mathrm{K}$$

The authors of this paper did not realize it, but they found the microwave background radiation. Their conclusion, that interstellar molecules seemed to be immersed in a bath of approx 3 K was exactly correct. Twenty some years later, two scientists at Bell Labs also discovered the background radiation, but were able (with the help of RH Dicke of Princeton University) to explain the origins of the bath as the remnant of the Big Bang.