PHYS 328
HOMEWORK 10-- SOLUTIONS

1. We start by considering the ratio of the probability of finding the system in the ionized state to the probability of finding the system in the state of a neutral H atom. This ratio is equivalent to the ratio of Gibbs factors. When the system is in the ionized state, there are no electrons so N = 0, and there is no energy so $\epsilon = 0$. When the system is in the neutral state, the chemical potential is μ and the energy is the ionization energy of the H atom which is equal to - I and N = 1. Therefore, the ratio of Gibbs factors is :

$$\text{prob}(p)/\text{prob}(H) = e^{-(0-0)/kT} \Big/ e^{-(-I-\mu)/kT} = e^{-I/kT}\, e^{-\mu/kT}$$

where p represents the proton (i.e., the ionized state) and H represents the neutral hydrogen atom. The ratio of probabilities is the same as the ratio of partial pressures, so we have :

$$\frac{P_p}{P_H} = e^{-I/kT}\, e^{-\mu/kT}$$

We are told in the problem to treat the electrons as an ideal gas. From chapter 6 (eq. 6.93) we have that the chemical potential for an ideal gas is :

$$\mu = -kT\ln\left(V Z_{int}/N v_q\right) \Rightarrow \mu/kT = -\ln\left(V Z_{int}/N v_q\right)$$

If we treat the electron as an ideal gas, the ideal gas law yields that V/N = k T/P_e where P_e is the electron pressure. Since an electron has no vibrational or rotational states, the value of Z_{int} is 1. The remaining term, v_q is the quantum volume given by eq. 6.83. Substituting these expressions into μ gives us:

$$\frac{P_P}{P_H} = e^{-I/kT}\, e^{\ln\left(kT/P_e v_q\right)} = e^{-I/kT}\cdot\frac{kT}{P_e v_q} = \frac{kT}{P_e}\left(2\pi m_e kT\big/h^2\right)^{3/2}$$

which is Saha' s equation as derived in equation 5.130 of the text.

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2. The system state corresponding to the proton (ionized atom) still has a Gibbs factor of 1. The neutral state now has two states, one for each spin state. Since the spin state does not effect either the ionization energy or the chemical potential, the Gibbs factor for the neutral state is two times its value in problem 1, so we have :

$$\frac{P_p}{P_H} = \frac{1}{2\, e^{-(-I-\mu)/kT}} \tag{1}$$

Now, remember that :

$$\frac{\mu}{kT} = -\ln\left(\frac{V}{N}\,\frac{Z_{int}}{v_q}\right)$$

There are now two spin states, each of which has a Z_{int} equal to 1, so that the value of Z_{int} for this problem is 2. Thus, we have

$$\frac{\mu}{k\,T} = -\ln(2\,k\,T / P_e)$$

Substitute this value of μ/k T into eq. (1) and you will see that the factors of 2 cancel resulting in the same expression in problem 1.

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3. a) If the particle can inhabit any of of 10 particle states, each of which has energy zero, the Boltzmann factor for each state is 0. The partition function is the sum of Boltzmann factors, so that the partition function for this system is Z = 10.

b) If there are two distinguishable particles, then

$$Z_{total} = Z_1\,Z_2 = 10 \cdot 10 = 100$$

c) If the particles are identical, there are 10 ·9/2 ways of putting the particles into two different states, and 10 ways of putting the particles in the same state (i.e., both particles in state 1, or state 2, ... state 10). Thus, the total number of possible ways of arranging the two particles is 55. Since the Boltzmann factor for each state is 1, the partition function for these two bosons is 55.

d) If the two particles are fermions, there are only 45 ways to arrange them (since you cannot put two fermions in the same state).

e) Eq. 7.16 leads to :

$$Z_{total} = \frac{Z_1^2}{2} = \frac{10^2}{2} = 50$$

f) If the particles are distinguishable, there are 100 total ways to arrange the particles, 10 of which have the particles in the same state, so the probability is 1/10. If the particles are bosons, the probability is 10/55. If the particles are fermions, the probability is zero since there can never be two fermions in the same state.

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4. Let' s first consider the ratio :

$$\frac{n_{BE}}{n_{FD}} = \frac{e^x + 1}{e^x - 1}$$

where x = (ϵ - μ)/kT. Now, we know that the two distributions approach each other when x becomes large (see Fig. 7.7 on p. 269). Dividing through by e^x we get:

$$\frac{n_{BE}}{n_{FD}} = \frac{1 + e^{-x}}{1 - e^{-x}} \approx 1 + 2\,e^{-x}$$

when x is large. If the ratio of the distributions is to be within 1 % of each other, then we have :

$$1 + 2\,e^{-x} < 1.01$$

$$\text{or } e^{-x} \approx \frac{1}{200} \text{ or } e^x \approx 200$$

Thus implies that x ≈ ln 200 or x ≈ 5.3

So if x is greater than about 5.3, the two distributions will lie within 1 % of each other. Now, the Boltzmann distribution lies between the Fermi - Dirac and the Bose - Einstein, so we already know that if the latter two are within 1 % of each other, so also is the Boltzmann. Under what conditions will this occur on the Earth?

Let' s estimate the value of x under terrestrial conditions. Since x = $(\epsilon - \mu)/k\,T$, the lowest value of x occurs when $\epsilon = 0$. If $\epsilon = 0$, then x = $-\mu/k\,T$. Remembering that μ is given by

$$\frac{\mu}{k\,T} = -\ln\left(\frac{V}{N}\cdot\frac{Z_{int}}{v_q}\right)$$

our condition (for the three distributions to approximate each other) becomes

$$\frac{V}{N}\cdot\frac{Z_{int}}{v_q} \approx 200 \text{ (by exponentiating both sides).}$$

Now, we know that V/N is k T/P (via the ideal gas law). Reviewing section 6.2 of the text reveals that Z_{int} for diatomic molecules in the Earth's atmosphere is about 50. (Remember, at room temps, the vibrational modes of diatomic molecules are frozen out, so only the rotational modes contribute to Z_{int}. Page 236 quotes a value of Z_{int} =100 for CO, but that the value for homonuclear molecules is roughly 1/2 of this (see eq. 6.33)). Thus, we will use a value of 50 for $Z_{int.}$ Now, using our definition of the quantum volume and substituting numbers for diatomic nitrogen, we find:

$$\frac{V}{N}\cdot\frac{Z_{int}}{v_q} = \frac{k\,T}{P}\cdot\frac{Z_{int}}{v_q} = \left(\frac{1.38\times10^{-23}\,\text{J/K} * 300\,\text{K} * 50}{10^5\,\text{Pa}}\right)$$

$$\left(\left(2\pi\left(28\cdot1.67\times10^{-27}\ \text{kg} * 1.38\times10^{-23}\,\text{J/K} * 300\,\text{K}\right)\right)\Big/\left(6.62\times10^{-34}\,\text{J}\cdot\text{s}\right)^2\right)^{3/2} \approx 3\times 10^8$$

What does this mean? Our analysis above showed that the three distributions would all be within 1 % of each other if x were at least as large as 200. Using parameters for the Earth' s atmosphere, we find that x is 300 million, so the condition is easily met. The moral of the story : at room temperature, the three distributions are indistinguishable from each other for atmospheric gases.

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5. Let' s grind and find :

The Fermi energy is :

$$\epsilon_F = \frac{h^2}{8\,m_e}\left(\frac{3}{\pi}\,\frac{N}{V}\right)^{2/3}$$

where m_e is the mass of an electron. For standard values of mass and density of copper, we get

$$V = \frac{\text{mass}}{\text{density}} = \frac{63.5\,\text{g/mole}}{8.93\,\text{g}/\text{cm}^3} = 7.11\,\text{g}/\text{cm}^3 = 7.11\times10^{-6}\,\text{m}^3$$

This is the volume of one mole of copper. If each atom contributes one electron, then one mole of copper contributes an Avogadro number of electrons and :

$$\frac{N}{V} = \frac{6\times 10^{23}}{7.1\times 10^{-6}\ \mathrm{m}^3} = 8.5\times 10^{28}\ \mathrm{m}^{-3}.$$

Substitute this into the equation for the Fermi energy and get :

$$\epsilon_F = 1.1\times 10^{-18}\ \mathrm{J}$$

The Fermi temperature is

$$T_F = \frac{\epsilon_F}{k} = \frac{1.1\times 10^{-18}\ \mathrm{J}}{1.38\times 10^{-23}\ \mathrm{J/K}} = 82\,000\ \mathrm{K}$$

Since room temp is 300 K, there is no problem in approximating the electrons in metals to be at 0 K.

The degeneracy pressure is

$$P = \frac{2}{5}\frac{N}{V}\epsilon_F = 3.8\times 10^{10}\ \mathrm{N}\big/\mathrm{m}^2$$

and the contribution to the bulk modulus is found from the equations :

$$B = \frac{10}{9}\frac{U}{V}\ \text{and}\ P = \frac{2}{3}\frac{U}{V}\ \text{so that}\ B = \frac{10}{9}\left(\frac{3}{2}P\right) = \frac{5}{3}P = 6.4\times 10^{10}\ \mathrm{Nm}^{-2}$$

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6. We start with the Planck spectrum :

$$u(\epsilon) = \frac{8\pi}{(h\,c)^3}\frac{\epsilon^3}{e^{\epsilon/k\,T}-1}$$

If we set $x = \epsilon/k\,T$, we get :

$$u(x) = \frac{8\pi}{(h\,c)^3}(k\,T)^3\frac{x^3}{e^x-1}$$

The peak in the distribution occurs when $d\,u(x)/dx = 0$. This yields :

$$\frac{d}{dx}u(x) = A\left[\frac{3\,x^2}{e^x-1} - \frac{x^3\,e^x}{(e^x-1)^2}\right] = 0$$

where A is the cluster of constants :

$$A = \frac{8\pi(k\,T)^3}{(h\,c)^3}$$

(the value of A does not matter since we want to solve the transcendental equation :

$$3\frac{x^2}{e^x-1} = \frac{x^3\,e^x}{(e^x-1)^2} \Rightarrow 3(e^x-1) = x\,e^x$$

and we want to find the value of x that satisfies this equation. We can plot the two functions :

In[2]:=
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Plot[{3 (Exp[x] - 1), x Exp[x]}, {x, 0, 3}]
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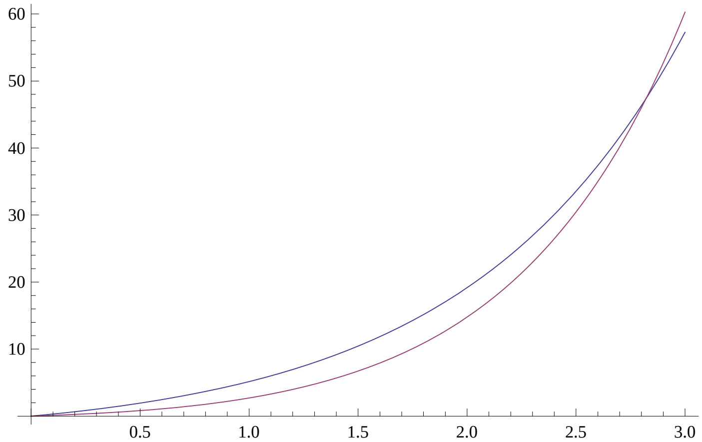
Out[2]=



and we can see there is a solution in the vicinit of 2.8. Using the FindRoot function :

In[5]:=
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FindRoot[3 (Exp[x] - 1) - x Exp[x], {x, 3}]
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Out[5]=
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{x → 2.82144}
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And we verify that the peak of the Planck distribution occurs at x = 2.82. Or that :

$$x = \frac{\epsilon}{k\,T} = 2.82 \text{ or}$$

Since ϵ is the energy of a photon, we know that $\epsilon = h\,\nu = h\,c/\lambda$ where λ is the wavelength of the photon. Thus, we can show that

$$\frac{h\,c}{\lambda\,k\,T} = 2.82 \text{ or } \lambda_{max} = \frac{h\,c}{2.82\,k\,T}$$

where λ_{max} is the wavelength corresponding to the peak of the Planck distribution. Recall that h, c and k are constants, so that we can write:

$$\lambda_{max} \propto \frac{1}{T}$$

which tells us that the hotter the radiating object, the shorter the wavelength at which its peak radiation is emitted.

7. Following the treatment on page 305, we can see that if the solar constant at the distance of the Earth is 1370 W/m^2, the solar constant at Venus will be greater by a factor of $(1/0.7)^2$ (since radiation varies as $1/r^2$). Thus, there is twice as much solar radiation (2740 W/m^2) at the top of the Venusian atmosphere compared to the Earth's. However, 77% of it is reflected to space as the result of the cloud layer of Venus, so that the amount of radiation penetrating the atmosphere is 1370 *2*0.23 =630 W/m^2.

If there were no atmosphere and thus no reflection, the temperature on the surface of Venus would be :

$$T = \left(2740\,\mathrm{W\,m^{-2}} / 4 \cdot 5.67\,10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}\right)^{1/4} = 331\,\mathrm{K}$$

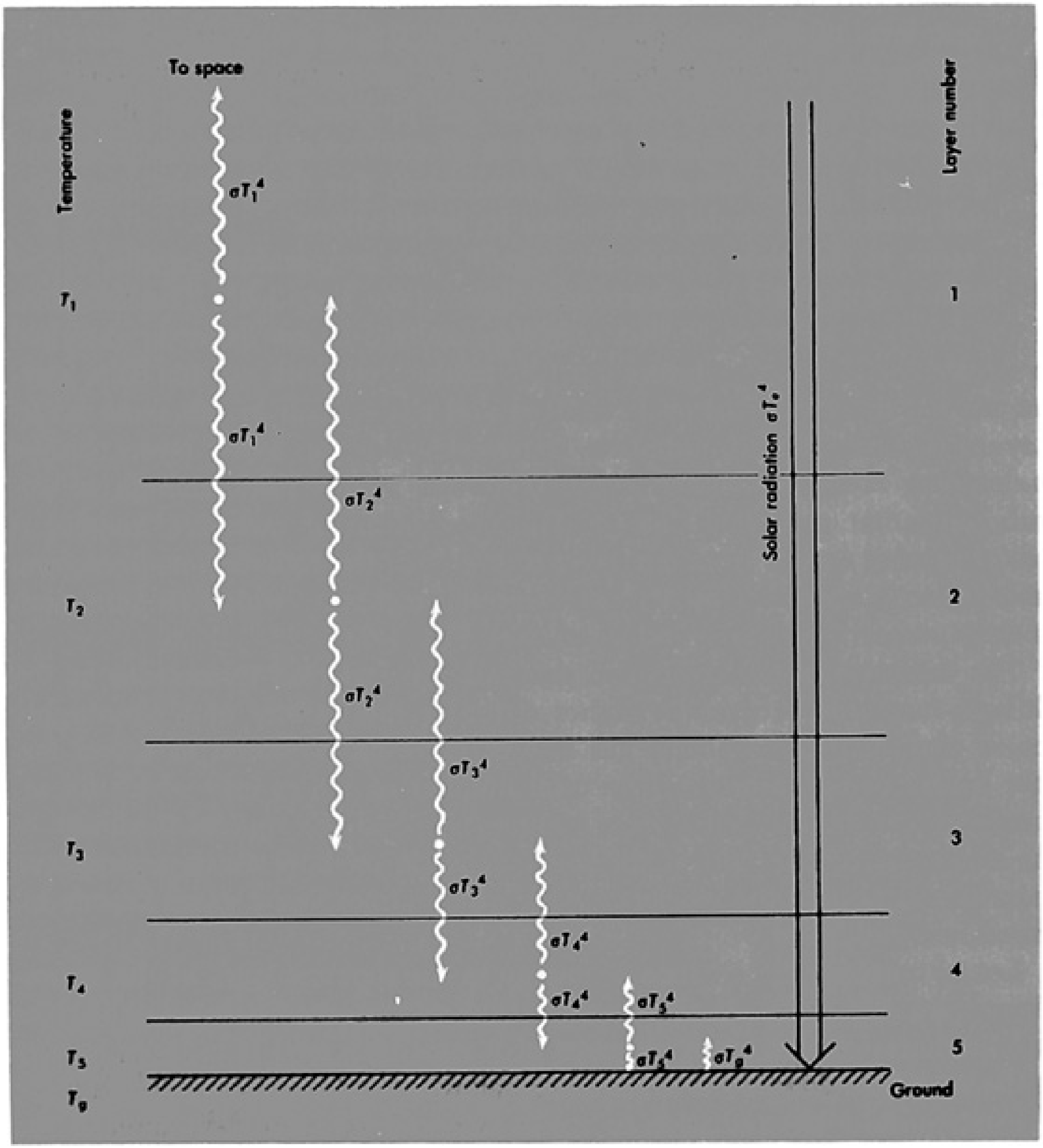
a bit warmer than Earth (but enough to evaporate carbon dioxide out of the calcium carbonate rocks on the primordial surface of Venus). With the atmosphere, Venus receives only 630/2740 = 0.23 of this energy, so the temperature at the surface (of a single layer atmosphere) is

$$T_{\text{surf}} = 0.23^{1/4}\, T_{\text{cloud top}} = 0.69 * 331\,\text{K} = 230\,\text{K}$$

c) However, the atmosphere of Venus is very thick and has a large optical depth (equivalent to the statement that we can approximate its atmosphere by 70 layers). As the graph below shows (cribbed from http : // lasp.colorado.edu/~bagenal/3720/CLASS14/14 EVM - 5. html), we can apply radiative equilibrium to each layer of the atmosphere. The top layer must be in equilibrium with the incoming solar radiation, so the top layer both absorbs and emits an amount of energy equal to $\sigma\, T_1^4$. You can see that this top layer emits this amount of energy in all directions, so that the second layer absorbs twice as much energy as the first, and therefore emits two units of energy upward and downward to the third layer. So, the second layer is emitting four units of energy, and receiving 1 from the upper layer; this means it must be receiving 3 units of energy from the next layer down. If that next layer is emitting 3 units of energy up and 3 units of energy down, it must be hotter than the first blanket by a factor of $3^{1/4}$(since it is emitting three times as much energy). So the third layer emits a total of 6 units of energy (3 up and 3 down), and receives 2 units from the second layer. Thus, the third layer is receiving 4 units of energy from the next layer down...and you can follow this all the way to the ground. The result you get is that the nth layer must be $n^{1/4}$ times hotter than the top layer. So going all the way to the ground (which represents the 71st layer), we expect the ground to be $71^{1/4}$ hotter than the top of the atmosphere, or the temperature at the surface of Venus is:

$$T_{\text{surf}} = 71^{1/4} * 230\,\text{K} = 667\,\text{K}!$$

And Omnec could not have survived those temperatures.


To space
Temperature
Layer number
T_1
T_2
T_3
T_4
T_5
T_g
1
2
3
4
5
σT_1^4
σT_1^4
σT_2^4
σT_2^4
σT_3^4
σT_3^4
σT_4^4
σT_4^4
σT_5^4
σT_5^4
σT_g^4
Solar radiation σT_e^4
Ground