PHYS 328
OPTIONAL HOMEWORK-- SOLUTIONS

1. The system has four different states : the state with no oxygen molecules, the state with one oxygen molecule bound (and there are two ways of achieving this state), and the state with two oxygen molecules. The energies of these states are, respectively, 0, -0.55 eV, -0.55 eV, and - 1.30 eV, so that the grand partition function is :

$$\mathcal{Z} = 1 + 2\,e^{-(-0.55-\mu)/kT} + e^{-(-1.30-2\mu)/kT}$$

the average number of oxygen molecules is :

$$\overline{N} = \Sigma_s N(s) = 0 \cdot P(0) + 1 \cdot N(1) + 2 \cdot N(2) = \frac{2}{\mathcal{Z}}\left[e^{-(-0.55-\mu)/kT} + e^{-(-1.30-2\mu)/kT}\right]$$

The occupancy is just half of this, so :

$$\text{occupancy} = \frac{1}{\mathcal{Z}}\left[e^{-(-0.55-\mu)/kT} + e^{-(-1.30-2\mu)/kT}\right]$$

By using the properties of exponentials, we can write the first exponential as :

$$e^{-(-0.55-\mu)/kT} = e^{-\epsilon_1/kT}\, e^{\mu/kT} = e^{-\epsilon_1/kT}\left(V Z_{int} / N v_Q\right)^{-1} = e^{-\epsilon_1/kT}\,\frac{P}{kT} \cdot \frac{v_Q}{Z_{int}}$$

where ϵ_1 is the energy of the first state (-0.55 eV),
P is the partial pressure of oxygen, and we have used the relationship

$$\mu = -kT \ln\left(\frac{V}{N}\,\frac{Z_{int}}{v_Q}\right)$$

We can write the second exponential in a similar form by writing ϵ_2, the energy of the second state in terms of the first state, so that we have:

$$\epsilon_2 = 2\,\epsilon_1 - \Delta \quad \text{where } \Delta = 0.2\text{ eV}$$

we can write the second exponential as

$$e^{-(2\epsilon_1 - 2\mu)/kT}\, e^{\Delta/kT} = e^{-((\epsilon_1-\mu)/kT)^2}\, e^{\Delta/kT}$$

While these steps involve a little bit of cumbersome algebra, they allow us to write both exponentials in terms of

$$e^{-\epsilon_1/kT}\,\frac{P}{kT} \cdot \frac{v_Q}{Z_{int}}$$

we can simplify our expressions by writing this term as ΓP where

$$\Gamma = e^{-\epsilon_1/kT}\,\frac{1}{kT} \cdot \frac{v_Q}{Z_{int}}$$

and we can write our occupancy in terms of Γ and P as :

$$\text{occupancy} = \frac{\Gamma P + e^{\Delta/k T} (\Gamma P)^2}{1 + 2 \Gamma P + e^{\Delta/k T} (\Gamma P)^2}$$

and we have an expression for occupancy in terms of the partial pressure of oxygen (P). Using standard values, we can compute and plot :

In[115]:=
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Clear[zint, vq, h, m, mp, k, T, Δ, e1, gamma, P]
h = 6.62 × 10^-34; k = 1.32 × 10^-23; mp = 1.66 × 10^-27; m = 32;
zint = 223; T = 310; Δ = 0.2 × 1.6 × 10^-19; e1 = -0.55 1.6 × 10^-19;

vq = (h / Sqrt[2 π m mp k T])^3;
gamma = Exp[-e1 / (k T)] vq / (k zint T);

occupancy[P_] :=
 (gamma P + Exp[Δ / (k T)] (gamma P)^2) / (1 + 2 gamma P + Exp[Δ / (k T)] (gamma P)^2)
Plot[occupancy[P], {P, 0, 20 000}, PlotRange → All]
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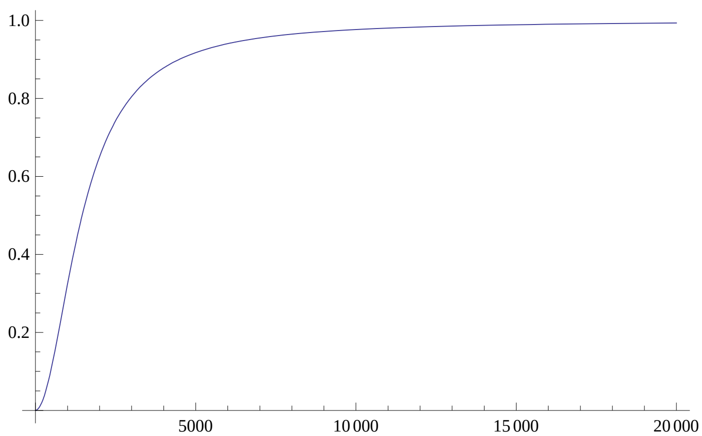
Out[120]=



Where the vertical axis is in occupancy number, and the horizontal axis is in units of Pa (up to a limit of 0.2 atm (since oxygen represents approximately 20 % of the Earth' s atmosphere)).

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2. We will be computing the occupancy number using the Fermi - Dirac distribution. In each case, we are given the value of (ϵ - μ) in eV. Since we are at room temperature, we know that k T = 0.025 eV. The Fermi - Dirac distribution can be written :

$$\bar{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/k T} + 1}$$

a) for (ϵ - μ) = -1 eV, we have

$$\bar{n}_{FD} = \frac{1}{e^{-1/(0.025)} + 1} = 1$$

b) for (ϵ - μ) = -0.01 :

$$\bar{n}_{FD} = \frac{1}{e^{-0.1/0.025} + 1} = \frac{1}{1.68} = 0.59$$

c) For $\epsilon = \mu$, the argument of the exponential is 0, so the exponent has a value of 1, and the fraction is 1/2.

d) For (ϵ - μ) = +0.01 :

$$\overline{n}_{FD} = \frac{1}{e^{0.1/0.025} + 1} = 0.41$$

e) For (ϵ - μ) = +1, the argument of the exponential is 40, so the denominator is dominated by the exponential, and the occupancy number goes as Exp[-40] which is essentially zero.

3. We showed in an earlier homework problem that the gravitational potential due to a self gravitating sphere of mass M and radius R is

$$\frac{3}{5} \frac{G M}{R^2}$$

b) The total kinetic energy of a degenerate electron gas (according to equations in the text) is given by :

$$U_K = \frac{3}{5} N \epsilon_F = \frac{3}{5} N \cdot \frac{h^2}{8 m_e} \left(\frac{3 N}{\pi V}\right)^{2/3}$$

where N is the number of electrons and m_e is the electron mass. The statement that the star contains one neutron and one proton for each electron allows us to write N, the number of electrons, in terms of the mass of the star. The mass of the star derives from the mass of protons and neutrons, and there are two nucleons per electron, so the number of electrons is :

$$N = \frac{M}{2 m_p}$$

where M is the mass of the star and m_p is the mass of the proton (since the masses of the proton and neutron are so similar, I can use m_p for both nucleons). Substitute this expression for N into the kinetic energy equation and get:

$$U_K = \frac{3 h^2}{40 m_e} \left(\frac{M}{2 m_p}\right)^{5/3} \left(\frac{9}{4 \pi^2 R^3}\right)^{2/3}$$

using the standard equation for the volume of a sphere. Evaluating all constants gives :

$$U_K = 0.0088 \frac{h^2 M^{5/3}}{m_e {m_p}^{5/3} R^2}$$

c) We know that gravitational potential energy goes as 1/R, and kinetic energy (as shown above) goes as $1 / R^2$. The equilibrium radius occurs when

$$\frac{d}{dR} \left(U_K + U_{pot}\right) = 0$$

Use the expressions in part b) and the previous homework result for potential energy, differentiate with respect to R, and find that

$$R_{eq} = \frac{2 \cdot 0.0088 \cdot h^2 M^{5/3} \big/ m_e m_p^{5/3}}{(3/5)\, G M^2}$$

substituting values for the mass of the sun $\left(2 \cdot 10^{30}\ \text{kg}\right)$ we find that $R_{eq} \approx 7200$ km

e) Substituting values into the expression for Fermi energy we get :

$$\epsilon_F = \frac{h^2}{8\, m_e} \left(\frac{3\,N}{\pi\, V}\right)^{2/3} = \frac{h^2}{8\, m_e} \left(\frac{9\, M}{8\, \pi^2\, m_p}\right)^{2/3} \cdot \frac{1}{R^2}$$

using the value of the solar mass for M and setting R = 7200 km (remember to use metric units), we get that

$$\epsilon_F = 3.1 \times 10^{-14}\ \text{J}$$

The Fermi temperature is then

$$T_F = \frac{\epsilon_F}{k} = \frac{3.1 \times 10^{-14}\ \text{J}}{1.38 \times 10^{-23}\ \text{J}/\text{K}} = 2.3 \times 10^9\ \text{K}$$

Since this is much hotter than the surface, or even the core of the sun, it is appropriate to assume a white dwarf is a degenerate electron gas of T = 0.

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4. a) The relevant equation here is :

$$\epsilon_0 = \frac{h^2}{8\, m\, L^2} \left(n_x^2 + n_y^2 + n_z^2\right)$$

For the ground state, all the energy levels have value 1; mass is the mass of a rubidium atom (which is 87 times the proton mass), and the L is the length of a side, which is given as 10^{-5}m. Substituting these values gives us $e_0 = 10^{-32}$ J = 7.1 10^{-14} eV .

b) Using eq. 7.126 in the text and using a value of N = 10,000, you should obtain that k $T_c = 10^{-30}$ J = 7.4 10^{-12}eV, so $T_c \sim 10^{-7}$K.

c) Use equation 7.129 :

$$N_0 = \left[1 - \left(\frac{T}{T_C}\right)^{3/2}\right] N$$

For $T/T_c = 0.9$, $N_0 = 0.146$ N. If N = 10, 000, there are 1460 atoms in the ground state.

Equation 7.120 gives :

$$N_0 = \frac{k\, T}{\epsilon_0 - \mu} \Rightarrow \epsilon_0 - \mu = \frac{k\, T}{N_0} = 0.9 * \left(7.4 \times 10^{-12}\ \text{eV}\right) \big/ 1460 = 4.6 \times 10^{-15}\ \text{eV}$$

or the chemical potential lies below the ground state.

d) Using $N = 10^6$ atoms, we get:

$$k\, T_c = 0.224\, N^{2/3}\, \epsilon_0 = 0.224 \left(10^6\right)^{2/3} \left(7.1 \times 10^{-14}\right) = 1.6 \times 10^{-10}\ \text{eV}$$

where 0.224 is the value of the cluster of constants in eq. 7.126 and we use the ground state energy from part a) since that does not change with the value of N. We can divide by the appropriate value of K to find the condesation temperature :

$$T_C = \frac{1.6\times 10^{-10}\ \text{eV}}{8.63\times 10^{-5}\ \text{eV/K}} = 1.85\times 10^{-6}\ \text{K}$$

When T = 0.9 T_c, equation 7.129 still gives us that 0.146 of the atoms are in the ground state, but now this is 14.6% of one million, meaning 146,000 atoms are in the ground state.

Now we solve for $\epsilon_0 - \mu$ for this case"

$$\epsilon_0 - \mu = 0.9\,(k\,T_c)/N_0 = 0.9 * 1.6\times 10^{-10}\ \text{eV} / 146\,000\ \text{atoms} = 9.83\times 10^{-16}\ \text{eV}$$

The chemical potential is still below the ground state energy, but now only by a much smaller percentage

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5. We use equation 7.126 in the form :

$$k\,T_c = 0.527\,\frac{h^2}{2\pi m}\left(\frac{N}{V}\right)^{2/3}$$

where m is the proton mass and N/V is the density of atoms. Converting the given density to MKS and substituting appropriate values, we find after dividing through by k :

$$T_C = 0.527\,\frac{\left(6.62\times 10^{-34}\ \text{Js}\right)^2}{2\pi\,1.67\times 10^{-27}\ \text{kg} * 1.38\times 10^{-23}\ \text{J/K}}\left(1.8\times 10^{20}\ \text{atoms}/\text{m}^3\right)^{2/3} = 5.1\times 10^{-5}\ \text{K}$$

which is very close to the measured value reported in the question.

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6. We are given the density of liquid helium and asked to treat it as a non - interacting gas, an assumption we should expect will lead to some errors.

We find the ratio N/V from :

$$\frac{N}{V} = \frac{0.145\ \text{g}/\text{cm}^3}{4\ \text{g/mol}} = 0.036\ \text{mol}/\text{cm}^3 = 3.6\times 10^4\ \text{mol}/\text{m}^3 = 2.2\times 10^{28}\ \text{atoms}/\text{m}^3$$

this becomes our value for N/V in eq. 7.126 and we find :

$$T_c = 0.527\,\frac{h^2}{2\pi m k}\left(\frac{N}{V}\right)^{2/3}$$

where m is now the mass of a He 4 nucles (4 times the proton mass) and N/V has the valued computed just above. Substituting these values gives us

T_C = 3.14 K

This is about 50 % higher than the actual value, but not horrible considering we are treating a liquid as a gas.